SELF–ADJOINT COMMUTING DIFFERENTIAL OPERATORS OF RANK 2 AND THEIR DEFORMATIONS GIVEN BY THE SOLITON EQUATIONS

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In [1] and [2] I.M. Krichever and S.P. Novikov introduced a remarkable class of exact solutions of soliton equations — solutions of rank $l > 1$. In this article we study solutions of rank two of the following system

$$V_t = \frac{1}{4}(6VV_x + 6W_x + V_{xxx}), \quad W_t = \frac{1}{2}(-3VW_x - W_{xxx}). \tag{1}$$

This system is equivalent to the commutativity condition of the self-adjoint operator $L_4 = (\partial_x^2 + V(x,t))^2 + W(x,t)$ and the skew-symmetric operator $\partial_t - \partial_x^3 - \frac{3}{2}V(x,t)\partial_x - \frac{3}{4}V_x(x,t)$. In this case "solutions of rank two" means that for every $t \in \mathbb{R}$ every operator commuting with L_4 has even order. It also means that the dimension of space of common eigenfunctions of commuting operators L_4 and L_{4g+2} is equal to two

$$dim_{\mathbb{C}} \{\psi : L_4\psi = z\psi, L_{4g+2}\psi = w\psi\} = 2$$

for generic eigenvalues (z, w). The set of eigenvalues $P = (z, w)$ forms hyperelliptic curve $w^2 = F_g(z) = z^{2g+1} + c_{2g}z^{2g} + \ldots + c_0$. This curve is called spectral.

There is a classification of commutative rings of ordinary differential operators of arbitrary rank obtained by Krichever [3] but in general case such operators are not found.

Krichever and Novikov [1] found operators of rank two corresponding to an elliptic spectral curve. Mokhov found operators of rank three corresponding to an elliptic spectral curve. In the case of spectral curves of genus 2, 3 and 4 it is known only examples of operators of rank greater than one.

Operators L_4 L_{4g+2} of rank two corresponding to hyperelliptic spectral curves were studied in [4]. Operators $L_4 - z$, $L_{4g+2} - w$ have common right divisor $L_2 = \partial_x^2 - \chi_1(x,P)\partial_x - \chi_0(x,P)$:

$$L_4 - z = \tilde{L}_2 L_2, \quad L_{4g+2} - w = \tilde{L}_{4g} L_2.$$

Functions $\chi_0(x,P)$, $\chi_1(x,P)$ are rational functions on Γ, they satisfy the Krichever's equations. The operator L_4 is self-adjoint if and only if $\chi_1(x,P) = \chi_1(x,\sigma(P))$ [4]. If $g \geq 1$, then the following theorem holds [4].

Theorem 1 [4]**.** *If L_4 is self-adjoint operator, then*

$$\chi_0 = -\frac{Q_{xx}}{2Q} + \frac{w}{Q} - V, \qquad \chi_1 = \frac{Q_x}{Q},$$

where $Q = z^g + \alpha_{g-1}(x)z^{g-1} + \ldots + \alpha_0(x)$. Polynomial Q satisfies equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q_x)^2 + (Q_{xx})^2 - 2Q_xQ_{xxx} + $$
$$+2Q(2V_xQ_x + 4VQ_{xx} + Q_{xxxx}). \tag{4}$$

The main aim of this paper is as follows. We study dynamics of polynomial Q provided that V and W satisfy (1).

Theorem 2. *Suppose that potentials V and W of operator $L_4 = (\partial_x^2 + V(x,t))^2 + W(x,t)$ commuting with operator L_{4g+2} satisfy the system (1). Then polynomial Q satisfies the following equation $Q_t = \frac{1}{2}\left(-3VQ_x - Q_{xxx}\right)$.*

Remark. *Similarly one can obtain the evolution equation on Q if in (2) one substitutes operator A by a skew-symmetric operator of order $2n + 1$. For example, in case of $n = 2, 3$.*

The following theorems are proved in [4] and [5].

Thank grants if you wish.

Theorem 3. *The operator*

$$L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + \alpha_3 g(g+1)x$$

commutes with an operator $L_{4g+2}^\sharp$ *of order* $4g+2$. *The spectral curve is given by the equation* $w^2 = F_{2g+1}(z)$, *where* F_{2g+1} *is a polynomial of degree* $2g+1$.

Theorem 4. *The operator*

$$L_4^\natural = (\partial_x^2 + \alpha_1 \cosh(x) + \alpha_0)^2 + \alpha_1 g(g+1)\cosh(x), \quad \alpha_1 \neq 0$$

commutes with an operator $L_{4g+2}^\natural$ *of order* $4g+2$. *The spectral curve is given by the equation* $w^2 = F_{2g+1}(z)$, *where* F_{2g+1} *is a polynomial of degree* $2g+1$.

The following theorems were proved in collaboration with E.I. Shamaev.

Theorem 5. *The operator* $L_4^\sharp$ *does not commute with any differential operator of odd order.*

Theorem 6. *The operator* $L_4^\natural$ *does not commute with any differential operator of odd order.*

Theorems 5 and 6 rigorously prove that $L_4^\natural$ from [4] and $L_4^\sharp$ from [6] are differential operators of rank two.

Theorem 7. *The operator* L *commuting with* $L_4^\natural$ *can be expressed algebraically in terms of* $L_4^\natural$ *and* $L_{4g+2}^\natural$.

References

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