ON INTEGRABILITY OF DISCRETE DYNAMICAL SYSTEMS IN THE REAL PLANE

SERGEI DUZHIN

In this talk, I do not prove any theorems. My main goal is to describe an example of a discrete dynamical system in the real plane having extraordinary properties. The results are obtained by computer experimentation. The initial idea was suggested by M.Kontsevich.

Let M be a smooth real manifold, $F : M \to M$ a smooth map, and $\{F^n\}_{n=1..\infty}$ the corresponding discrete dynamical system. We say that it is integrable if there is a smooth function $I : M \to \mathbb{R}$ such that $I \circ F = I$, that is, if the trajectory $\{F^n(x)\}_{n=1..\infty}$ of any point lies entirely on a certain level surface of the function I. If the mapping F is periodic, then it is certainly integrable.

We will discuss a particular case of this general situation where $M = \mathbb{R}^2_+$, $F(x,y) = (y, f(y)/x)$, and $f(y)$ is a Laurent polynomial in y with positive coefficients. We will give examples of both integrable and non-integrable systems of this kind.

Then we will study in more detail this dynamical system with $f(y) = y + y^{-1}$ and see that it is non-integrable but very close to integrable: its trajectories lie on smooth curves with an error less than 10^{-30}. It is likewise remarkable that the mapping F itself has only one fixed point in $\mathbb{R}^2_+$, all its iterates F^n with $1 \leq n \leq 85$ also have only one fixed point, but F^{86} has an extra 172 fixed points arranged along a convex curve.

Steklov Institute of Mathematics, St.Petersburg Department, Fontanka 27, St.Petersburg, 191023 Russia

E-mail address: duzhin@pdmi.ras.ru

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