ZETA FUNCTIONS OF MATRICES

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Let G be a group and $\mathbb{Z}G$ be the corresponding group ring. Any element $a \in \mathbb{Z}G$ can be uniquely written as a finite sum of the form

$$a = \sum_{g \in G} (a, g)\, g\,,$$

where (a, g) are integers. We define a linear map $\tau_1 : \mathbb{Z}G \to \mathbb{Z}$ by $\tau_1(a) = (a, e)$, where e is the identity element of G. It is easy to check that τ_1 is a trace map, i.e., it satisfies the identity $\tau_1(ab) = \tau_1(ba)$ for all $a, b \in \mathbb{Z}G$.

This trace map extends to a trace map $\tau_d : M_d(\mathbb{Z}G) \to \mathbb{Z}$ on the algebra $M_d(\mathbb{Z}G)$ of $d \times d$-matrices with entries in $\mathbb{Z}G$. For any matrix $M = (m_{i,j})_{1 \leq i,j \leq d} \in M_d(\mathbb{Z}G)$, the integer $\tau_d(M)$ is defined by

$$\tau_d(M) = \tau_1\left(\mathrm{Tr}(M)\right) = \sum_{i=1}^{d} \tau_1(m_{i,i})\,.$$

This definition allows us to associate to any matrix $M \in M_d(\mathbb{Z}G)$ the following formal power series in one variable t and rational coefficients:

$$Z_M = \exp\left(\sum_{n \geq 1} \tau_d(M^n)\, \frac{t^n}{n} \right) \in \mathbb{Q}[[t]]\,. \tag{1}$$

By analogy with other classes of zeta functions, we call Z_M the *zeta function* of the matrix M.

When the group $G = \{e\}$ is trivial and M is a square matrix with integer entries (more generally, with entries in a commutative ring), it is easy to check that

$$Z_M = \frac{1}{\det(1 - tM)}$$

is the inverse of the reciprocal polynomial of the characteristic polynomial of M (which captures the non-zero eigenvalues of M). This elementary fact motivates Definition (1).

When G is not trivial, there is no reason why Z_M should be a rational function. Nevertheless, we have the following theorem.

Theorem. *If G is a free group, then for each matrix $M \in M_d(\mathbb{Z}G)$ the zeta function Z_M has integer coefficients and is an algebraic function.*

"Algebraic" means that $y = Z_M$ satisfies a polynomial equation of the form

$$y^r + a_1(t) y^{r-1} + \cdots + a_r(t) = 0\,,$$

where $a_1(t), \ldots, a_r(t)$ are rational functions.

The special case $d = 1$ of the theorem is due to Kontsevich [2]; the general case $d \geq 1$ was treated in [3] by C. Reutenauer and the speaker. The algebraicity result for Z_M also holds for any *virtually free group* G, i.e. for any group containing a free group as a subgroup of finite index, but it does not hold in general: for instance, it does not hold for free abelian groups.

An example. In [3] the zeta function for the following matrix with entries in the free group group on three generators a, b, d was computed:

$$M = \begin{pmatrix} a + a^{-1} & b \\ b^{-1} & d + d^{-1} \end{pmatrix}.$$

We obtained

$$\begin{aligned} Z_M &= \frac{(1-8t^2)^{3/2}-1+12t^2-24t^4}{32t^6} \\ &= 1+3\,t^2+12\,t^4+56\,t^6+288\,t^8+1584\,t^{10}+4576\,t^{12}+\text{higher order terms;} \end{aligned}$$

it satisfies the polynomial equation

$$y^2+\frac{24t^4-12t^2+1}{16t^6}y+\frac{9t^2-1}{16t^6}=0\,.$$

About the proof of the theorem. (a) The fact that the formal power series Z_M has integer coefficients follows from a "Eulerian factorization" of Z_M of the form

$$Z_M=\prod_{w\in L}\frac{1}{1-a_w\,t^{|w|}}\,,$$

where the infinite product runs over the set of Lyndon words w in some alphabet and the coefficients a_w are integers.

(b) The algebraicity of Z_M implies the algebraicity of its logarithmic derivative

$$g_M=t\,\frac{d\log(Z_M)}{dt}=\sum_{n\geq 1}\tau_d(M^n)\,t^n\,.$$

The algebraicity of g_M had been proved several times before, in connection with language theory, with von Neumann algebras, or with free probabilities (see for instance [1, 4]). To conclude the proof of the theorem, we need the following result.

Lemma. *If $f\in\mathbb{Z}[[t]]$ is a formal power series with integer coefficients such that $t\,d\log f/dt$ is algebraic, then f is an algebraic function.*

To my knowledge there is no elementary proof of this lemma; one has to use deep techniques of arithmetic geometry.

References

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