GEOMETRIC REPRESENTATIONS OF FUNDAMENTAL GROUPS TO THE SYMMETRIC GROUP

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Let M be a n-dimensional manifold, $n \geqslant 2$, and let T be a triangulation of M. In the talk [1] Rubinstein introduced the method for constructing a representation from $\pi_1(M)$ to symmetric group S_{n+1}. This representation significantly depends on the choice of triangulation T.

We present an analogue of such representations for fundamental groups of 3-manifolds, which defines by *even special spines* (i.e. special spines, where each 2-component is a polygon with even number of sides). Such representations of $\pi_1(M)$ to S_4 we call *geometric*. If T is a triangulation of a 3-manifold M and P is an even special spine of M dual to T, then Rubinstein representation and geometric representation of $\pi_1(M)$ to S_4 are coincide.

Let $C = \{0, 1, 2, 3\}$, and let v be a true vertex of a special polyhedron P. By *coloring of a vertex* v we mean a bijection between arcs of edges of a special graph $S(P)$, which incident to v, and elements of C. Let e be an edge of a graph $S(P)$, which connect true vertices u and v of a polyhedron P. Define a *transfer* of a coloring of vertex v along edge e by the following way: we color an arc of the edge e by the same color as in the opposite side. All other three arcs near vertex u we color in a such way that in the neighborhood of the edge e in polyhedron P each 2-component incident to arcs of exactly two different colors. It can be verified that transfer of colors correctly define a representation of the group $\pi_1(P, v)$ to S_4. Such presentations are called *geometric*.

There are close relations between geometric representations and crystallization theory. Recall that *gem* is a regular graph of degree 4 with edges colored by 4 colors such that at each vertex all 4 colors meet together (also see [2]). Any gem generate a special polyhedron in the natural way: we attach 2-cell along each 2-cycle of the gem.

Theorem. *Let M be a 3-manifold, P be an even special spine of M. Then geometric representation of $\pi_1(M)$ is trivial if and only if polyhedron P generated by any gem.*

Corollary.

1. *Any 3-manifold (maybe after removing some 3-balls) admit an even special spine;*
2. *Let M be a closed 3-manifold, and let any representation of $\pi_1(M)$ to S_4 is trivial. Then M does not admit an even special spine;*
3. *If closed 3-manifold M admit an even special spine, then corresponding geometric representation of $\pi_1(M)$ to S_4 is not trivial.*

References

[1] J. Hyam Rubinstein, "Triangulations of n-Manifolds", *Mathematisches Forschungsinstitut Oberwolfach*, Report No. 24, 28-32 (2012).

[2] P. Bandieri, M. R. Casali, C. Gagliardi, "Representing manifolds by crystallization theory: foundations, improvements and related results", *Atti Sem. Mat. Fis. Univ. Modena Suppl.*, 49, 283–337 (2001).

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