THE GENERALISED KUPERBERG BRACKET AND MINIMALITY PROBLEM

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This work is based on a paper by V.O. Manturov and the author [3].

In the present work, we combine the ideas of the papers [1] and [2] and construct the "virtual" Kuperberg bracket for classical two-component links $L = J \sqcup K$ with one component (J) fibred (a definition of the fibred knot see, e.g., in [1]). Can be shown that this links can be using by virtual methods (see [1]). We consider a new geometrical complexity for such links and establish minimality of diagrams in a strong sense.

Definition 1. *A virtual diagram (or a diagram of a virtual link)* is the image of an immersion of a framed 4-valent graph in $\mathbb{R}^2$ with a finite number of intersections of edges.

Definition 2. *A virtual link* is an equivalence class of virtual diagrams modulo Reidemeister moves (see Fig. 1).

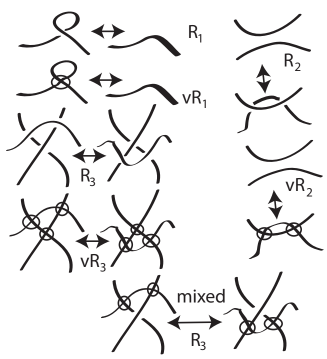

Figure 1. The Reidemeister moves

Let $\mathcal{S}$ be the collection of all connected trivalent bipartite oriented graphs such that each trivalent vertex of it has either three inward-oriented edges or three outward-oriented edges. Moreover, all graphs in $\mathcal{S}$ are finite, but loops and multiple edges are allowed. Let $\mathcal{T} = \{t_1, t_2, ...\}$ be the (infinite) subset of connected graphs from $\mathcal{S}$ having neither bigons nor quadrilaterals. We call this graph having neither bigons nor quadrilaterals *a irreducible graph*. Otherwise, we call a graph *reducible*.

Let $\mathcal{M}$ be the module $\mathbb{Z}[A, A^{-1}][t_1, t_2, ...]$ of formal commutative products of graphs from $\mathcal{T}$ with coefficients that are Laurent polynomials. Besides, let $\mathcal{D}$ be the collection of all diagrams of virtual links.

The proof of the next theorem see in in the work [2].

Theorem 1. There is unique map $f : \mathcal{D} \to \mathcal{M}$ which satisfies the relations $(I) - (VI)$ in Figure 2.

We define *the bracket* $[[\cdot]]$ *(generalised Kuperberg bracket)* as follows.

Let L be an oriented virtual diagram. We can define a polynomial $[[L]]$ as follows:

$$[[L]] = A^{-8w(L)} \cdot \ll L \gg,$$

where $w(L)$ be *a writhe number* of the diagram L (see, e.g., [5]).

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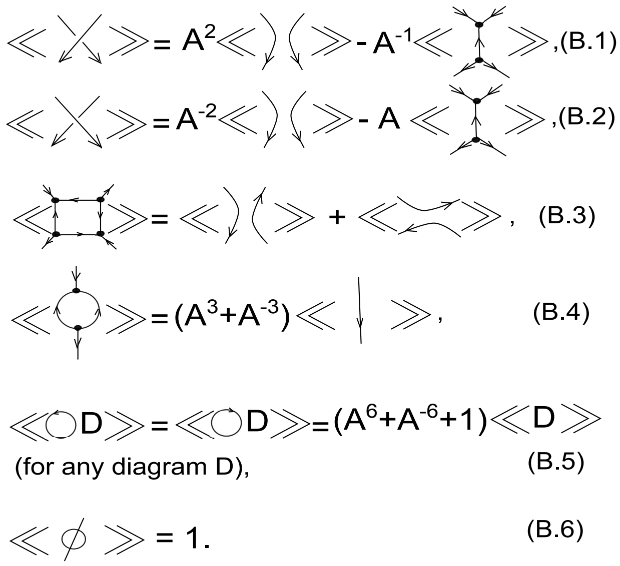

Figure 2. Skein relations for the Kuperberg bracket

The following theorem holds [2].

Theorem 2. The bracket [[.]] is invariant under all Reidemeister moves.

The generalised Kuperberg bracket can be application to establish minimality of some diagrams of classical links. Namely, we find a knot diagram K having a minimum number of double points of the projection on the Seifert surface of the fibred knot J.

Let $L = J \sqcup K$ ($lk(L) = 0$) be a classical two-components links with one component (J) fibred.

Definition 4. Let J be a fibred knot and Σ be an unique (see [6]) Seifert surface of J. Let K be a knot diagram on Σ. We will say that K is *minimal with respect to (w.r.t.)* J if for all K' is ambient isotopic to K in $\Sigma \times \mathbb{R}$, we have that the number of crossings of the diagram of K on Σ is less than or equal to the number of crossings of K' of the projection on Σ.

By the *unoriented state* K_{us} (K_{us}*-graph*) of K we mean the state of K where all crossings are resolved in a way where edge is added (see Fig. 2).

The following theorem is *a sufficient condition of minimality* of the diagram of the knot in thickened surface and is the *main result* of the present work. As it was noted above, this condition apply for any virtual diagrams unlike result of the work [1] which works only for irreducible odd diagram [4].

Theorem 3. Let $L = J \sqcup K$ be a classical two component link such that $lk(L) = 0$, where J is a fibred knot. Furthermore let as before Σ be a Seifert surface of the fibred knot J, and $\hat{K}$ be a virtual diagram corresponding to the knot K' in $\Sigma \times \mathbb{R}$. Then if the graph K_{us} of the diagram of $\hat{K}$ is irreducible then K' has a minimal number of double points of the projection on Σ, i.e. $\hat{K}$ is minimal w.r.t. J.

Theorem 3 is proved by contradiction using the result of Theorem 2.

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