KNOT INVARIANTS AND THEIR GRAPHICAL CALCULUS VIA HOWE DUALITY

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It is a well understood story that one can extract link invariants from quantum groups associated to simple Lie algebras. These invariants are called Reshetikhin-Turaev invariants and the famous Jones polynomial is the simplest example. Kauffman showed that the Jones polynomial could be described by a very simply an intuitive graphical calculus by replacing crossings in a knot diagrams by various smoothings. This approach can be generalized to provide graphical descriptions of link invariants associated to other Lie algebras.

More recently, Khovanov has shown that these quantum link invariants are shadows of a richer and more robust theory. Khovanov associated to link or knot a bigraded homology theory. The Jones polynomial is then recovered as the graded Euler characteristic of this homology. This new invariant is a stronger knot invariant, and is also functorial with respect to link cobordisms. The philosophy of uncovering richer structures that simplify to a known mathematical object is called categorification. Khovanov's categorification of the Jones polynomial ushered in a new area of link homology theories categorifying quantum link invariants.

Quantum invariants derive their name because they can be understood using the representation theory of certain Hopf algebras called quantum groups. These objects arise from deformations of universal enveloping algebras of semi-simple Lie algebras.

One of the original motivations for categorifying quantum groups was to provide a representation theoretic explanation for the existence of Khovanov homology and other link homologies categorifying quantum link invariants. Just as the Jones polynomial is described representation theoretically by the quantum group $U_q(\mathfrak{sl}_2)$ and tensor powers of its two dimensional representation, the categorification of the Jones polynomial via Khovanov homology should be described in terms of the 2-representation theory of the categorified quantum group associated to $U_q(\mathfrak{sl}_2)$.

Currently, the link between categorified quantum groups and Khovanov homology follows the indirect path through Webster's work on categorified tensor products. This connection utilizes an isomorphism relating Webster's categorifications of tensor products with categories associated to blocks of parabolic graded category $\mathcal{O}$. Categorifications associated with category $\mathcal{O}$ were initiated by Bernstein, Frenkel, and Khovanov and were further developed Stroppel. The relation to the familiar picture-world of Khovanov homology then relies on several technical results of Stroppel relating the knot homologies constructed using category $\mathcal{O}$ to Khovanov's more elementary construction. More generally, for link homology theories associated with fundamental $\mathfrak{sl}_n$ representations, Webster describes an isomorphism relating his construction to Sussan's category $\mathcal{O}$ based link homology theory, which is related via Koszul duality to a theory defined by Mazorchuk and Stroppel. When $n = 3$, the latter of these link homologies can then be identified with Khovanov's more elementary construction of $\mathfrak{sl}_3$ link homology defined using singular cobordisms called foams.

Alternatively, there is an algebro-geometric construction of Khovanov homology and related $\mathfrak{sl}_n$ link homologies due to Cautis and Kamnitzer. These knot homologies arise from derived categories of coherent sheaves on algebraic varieties associated to orbits in the affine Grassmannian. In the $\mathfrak{sl}_2$ case this knot homology agrees with Khovanov homology and these geometric categories can be understood as 2-representations of categorified quantum groups. These link homologies are related to those of Seidel-Smith and Manolescu by mirror symmetry.

In this talk, we provide a new bridge between quantum invariants and quantum groups using a duality called skew Howe duality. This duality allows us to directly translate relations in the quantum group into pictorial relations in the skein theory for the corresponding link invariant. In particular, we will show how to rediscover Kauffman's graphical calculus for the Jones polynomial and their generalizations to $\mathfrak{sl}_n$ using skew-Howe duality.

We then categorify this process to provide a direct construction of foam based $\mathfrak{sl}_n$ link homology theories intrinsically in terms of categorified quantum groups. We show that all of the components involved in these knot homologies are already present within the structure of categorified quantum groups including the relations in foam categories and the complexes defining the braiding. Utilizing Cautis-Rozansky categorified clasps we also obtain categorified projectors lifting Jones-Wenzl idempotents and their $\mathfrak{sl}_n$ analogs purely from the higher relations of categorified quantum groups. In the $\mathfrak{sl}_2$ case this work reveals the importance of a modified class of foams introduced by Christian Blanchet, suggesting that this version of the foam category is most natural from the perspective of categorified quantum groups. In the $\mathfrak{sl}_3$ case these results suggest a similar modified version of the $\mathfrak{sl}_3$ foam category.

Title 1: Knot invariants and diagrammatic descriptions of representation categories via Howe duality

Abstract: It is a well understood story that one can extract link invariants from quantum groups associated to simple Lie algebras. These invariants are called Reshetikhin-Turaev invariants and the famous Jones polynomial is the simplest example. Kauffman showed that the Jones polynomial could be described very simply by replacing crossings in a knot diagrams by various smoothings. In this talk we will explain Cautis-Kamnitzer-Licata's simple new approach to understanding these invariants using basic representation theory and the quantum Weyl group action. Their approach is based on a version of Howe duality for exterior algebras called skew-Howe duality. Even the graphical (or skein theory) description of these invariants can be recovered in an elementary way from this data. The advantage of this approach is that it suggests a 'categorification' where knot homology theories arise in an elementary way from higher representation theory and the structure of categorified quantum groups.

Title 2: Categorified knot invariants from categorified Howe duality

Abstract: Traditional representation theory of Lie algebras studies actions of the Lie algebra on vector spaces. Categorical representation theory studies actions of Lie algebras on categories, with Lie algebra generators acting by functors, and equations between elements lifting to isomorphisms of functors. Categorified quantum groups govern what kinds of natural transformations one can expect between these functors. It turns out that this higher structure can be encoded in a convenient graphical calculus. We will explain how categorified quantum groups and a categorification of the quantum Weyl group action can be used to categorify quantum link invariants.

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