THE HABIRO RING AND UNIFIED QUANTUM (WITTEN-RESHETIKHIN-TURAEV) INVARIANTS OF 3-MANIFOLDS

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For a ring R, $\mathbb{Z} \subset R \subset \mathbb{Q}$ let

$$\widehat{R[q]} = \varprojlim_n R[q]/((q;q)_n,$$

where

$$(q;q)_n = \prod_{j=1}^{n} (1-q^j).$$

Every element $f \in \widehat{R[q]}$ can be written as an infinite sum

$$f = \sum_{n=0}^{\infty} f_n(q)(q;q)_n, \quad f_n(q) \in R[q].$$

If $\xi \in \mathbb{C}$ is a complex root of 1, then $(\xi;\xi)_n = 0$ whenever $n \geq$ the order of ξ. Hence, one can define $f(\xi) \in R[\xi] \subset \mathbb{C}$ for every $\xi \in \mathcal{Z}$, the set of roots of 1.

The ring $\widehat{\mathbb{Z}[q]}$ (with $R = \mathbb{Z}$) is called the Habiro ring. It turns out that the functions $f \in \widehat{\mathbb{Z}[q]}$ with domains the set $\mathcal{Z}$ of roots of unity have properties similar to those of analytic functions. Every function $f \in \widehat{\mathbb{Z}[q]}$ has a natural Taylor series expansion at every element of the domain $\mathcal{Z}$, and two functions $f, g \in \widehat{\mathbb{Z}[q]}$ are the same if and only their Taylor expansions are the same at some element of $\mathcal{Z}$. Besides, two functions $f, g \in \widehat{\mathbb{Z}[q]}$ are the same if they agree on the subset of $\mathcal{Z}$ which has a limit in the "cyclotomic" topology.

The Habiro ring $\widehat{\mathbb{Z}[q]}$ is an integral domain, while $\widehat{R[q]}$, with $R = \mathbb{Z}[1/d]$ (with $d > 1$) or $R = \mathbb{Q}$, is not an integral domain.

Suppose $\mathfrak{g}$ is a simple Lie algebra. There is a subset $\mathcal{Z}_\mathfrak{g}$ of roots of unity for which one can define the Witten-Reshetikhin-Turaev invariant $\tau_M^\mathfrak{g}(\xi)$ of closed oriented 3-manifolds. The construction uses the theory of quantized universal enveloping algebras. In joint work with Habiro we show that if M is an integral homology 3-sphere, there is an invariant $J_M \in \widehat{\mathbb{Z}[q]}$ such that for every $\xi \in \mathcal{Z}_\mathfrak{g}$, one has $J_M(\xi) = \tau_M^\mathfrak{g}(\xi)$. This generalizes a result of Habiro for the case $\mathfrak{g} = sl_2$. As a corollary, we show that the WRT invariant of integral homology 3-spheres can be recovered from the LMO invariant. Another important corollary is the integrality result: if M is an integral homology 3-sphere, then $\tau_M^\mathfrak{g}(\xi)$ is always an algebraic integer, actually, it belongs to $\mathbb{Z}[\xi]$. The conjecture that $\tau_M^\mathfrak{g}(\xi)$ is an algebraic integer for every 3-manifold and every root of 1 is still open, except for the case $\mathfrak{g} = sl_2$.

When M is a rational homology 3-sphere, with $|H_1(M,\mathbb{Z})| = d$, we (with Bühler and Beliakova) show that there is an invariant $J_M \in \widehat{R[q]}$ with $R = \mathbb{Z}[1/d]$ such that $J_M(\xi) = \tau_M^{SO(3)}(\xi)$. Here $\tau_M^{SO(3)}(\xi)$ is a version of the WRT invariant for the group $SO(3)$.

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