COUNTING GRAPH CONFIGURATIONS IN 3-MANIFOLDS

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In the first talk, we will explain how to count graph configurations in 3-manifolds in order to obtain invariants of knots, links and 3-manifolds, following Gauss (1833), and, more recently, Witten, Bar-Natan, Kontsevich [Kon94], Poirier, Kuperberg, Thurston [KT99] and others. We will warm up with several equivalent definitions of the simplest of these invariants, which is the Gauss linking number of two-component links, pursue with a definition of the Casson-Walker invariant of rational homology 3-spheres (closed oriented 3-manifolds with the same rational homlogy as S^3) as an algebraic count of configurations of the θ-graph, and show how these definitions generalize to produce many more powerful invariants of links in rational homology 3-spheres, which count configurations of other unitrivalent graphs Γ, as outlined below.

let R be a rational homology 3-sphere. Embed the complement $S^3 \setminus B_1$ of the unit ball B_1 of $\mathbb{R}^3$ in ($S^3 = \mathbb{R}^3 \cup \infty$) into R and call ∞ the image of ∞ under this embedding, which identifies the neighborhood of ∞ in R with a neighborhood of ∞ in S^3. Let $\check{R} = R \setminus \{\infty\}$.

Lemma. *There is a natural compactification $C_2(R)$ of ($\check{R}^2 \setminus$ diagonal) such that:*

- *there is a homotopy equivalence $p_{S^2}\colon C_2(S^3) \to S^2$ that maps $(x,y) \in (\mathbb{R}^3)^2 \setminus$ diagonal to $\frac{y-x}{\|y-x\|}$,*
- *$C_2(R)$ is a smooth oriented 6-manifold with corners,*
- *$C_2(R)$ is a rational homology S^2 (i.e. $H_*(C_2(R);\mathbb{Q}) \cong H_*(S^2;\mathbb{Q})$),*
- *a parallelization τ of $\check{R}$ that coincides with the standard parallelization τ_s of $\mathbb{R}^3$ on $\mathbb{R}^3 \setminus B_1$ induces a natural map $p_\tau\colon \partial C_2(R) \to S^2$, such that $p_{\tau_s} = p_{S^2|\partial C_2(S^3)}$, and $p_{\tau|\partial C_2(R)\cap C_2(\mathbb{R}^3\setminus B_1)} = p_{\tau_s}$.*

Given (R,τ) as above, a *propagator* of (R,τ) is a 4-dimensional chain of $C_2(R)$ whose boundary is $p_\tau^{-1}(a)$ for some a in S^2. The first example of such a propagator for (S^3,τ_s) is $p_{S^2}^{-1}(a)$. In general, any 2-component link $J\coprod K\colon S^1\coprod S^1 \hookrightarrow \check{R}$ provides an embedding $J\times K\colon S^1\times S^1 \hookrightarrow C_2(R)$, and the algebraic intersection of $J\times K$ with a propagator in $C_2(R)$ is the linking number of J and K, so that a propagator can be thought of as a representative of the linking form.

The algebraic intersection of three transverse propagators of (R,τ) in $C_2(R)$ is an invariant of (R,τ), which is denoted by $\Theta(R,\tau)$ and called the Θ-invariant. For an integer homology 3-sphere H, relative Pontrjagin classes select a preferred parallelization τ_0, up to homotopy. In this case, according to a theorem of Kuperberg and Thurston [KT99], $\Theta(H,\tau_0) = 6\lambda(H)$ where λ is the Casson invariant.

More generally, generic collections of propagators allow us to count trivalent graphs with oriented edges decorated by propagators as follows. For such a trivalent graph Γ, we can algebraically count the injections of the set of vertices of Γ to $\check{R}$ that restrict to the ordered pair of vertices of every edge as an element of the associated propagator, as an algebraic intersection, too.

Suitable combinations of these algebraic intersection numbers provide invariants of rational homology 3-spheres. According to a theorem of Kuperberg and Thurston [KT99], to its interpretation in [Les04a, Les04b, Les13b] and to the classification of finite type invariants of integer homology 3-spheres by Le [Le97], any finite type invariant of integer homology 3-spheres can be written as such a suitable combination. Le obtained his classification of finite type invariants of integer homology 3-spheres with the help of the Le-Murakami-Ohtsuki universal finite type

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invariant Z_{LMO}, which is valued in a graded space generated by trivalent diagrams and quotiented by some relations that are called antisymmetry and Jacobi relations in analogy with the similar relations in Lie algebras. Counting graphs in configuration spaces provides an invariant Z_{KKT}, valued in the same space where the coefficient of a trivalent diagram is obtained by counting its configurations as outlined above.

Results of Massuyeau [Mas13], Moussard [Mou12] and [Les04b] show that Z_{LMO} and Z_{KKT} also share universality properties that prove that they distinguish exactly the same rational homology 3-spheres.

Kricker constructed a lift $\tilde{Z}$ of the Kontsevich(-LMO) integral of null-homologous knots in rational homology 3-spheres valued in a space of trivalent diagrams with edges decorated by rational functions of t with a denominator that divides the Alexander polynomial [GK04]. Such a lift organizes finite type knot invariants in an efficient way and shows how they behave under branched covering. In [Les10, Les11], we introduce equivariant propagators and we give an equivariant construction of an invariant $\tilde{Z}_{conf}$ valued in the same space, which is conjecturally equivalent to this Kricker lift. In [Les13a], we prove this conjecture for knots with trivial Alexander polynomial by proving a universality property for $\tilde{Z}_{conf}$ that generalizes a universality property proved by Garoufalidis and Rozansky in [GR04] for $\tilde{Z}$.

In the second talk, we will associate explicit propagators with Morse functions of 3-manifolds as in [Les12] in order to interpret the invariant Θ and its generalizations as counts of configurations of graphs where the edges follow flow lines of Morse functions. We will also show that the Θ-invariant is actually an invariant of rational homology 3-spheres equipped with a nowhere vanishing vector field, and we will show how to compute Θ from a Heegaard diagram [Les14].

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