FIXED POINT THEOREMS AND ITS DISCRETE ANALOGS

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In this talk we consider extensions of the classic Brouwer, Borsuk–Ulam, KKM and Kakutani fixed point theorems as well as its discrete analogs Sperner, Tucker and Ky Fan lemmas [1, 2, 3, 4, 5].

Let M^n be a closed PL manifold with a free simplicial involution $T : M \to M$, i.e. $T^2(x) = x$ and $T(x) \neq x$ for all $x \in M$. We say that a map $f : M \to \mathbb{R}^d$ is *antipodal* (or equivariant) if $f(T(x)) = -f(x)$. We call a pair (M, T) as a *BUT (Borsuk-Ulam Type) manifold* if for if for any continuous antipodal $f : M^n \to \mathbb{R}^n$ the zeros set $Z_f := f^{-1}(0)$ is not empty. We proved the following theorem:

Theorem 1. *Let M^n be a closed connected manifold with a free involution T. Then the following statements are equivalent:*
(a) M is BUT.
(b) M admits an antipodal continuous transversal map $h : M^n \to \mathbb{R}^n$ with $|Z_h| = 4k+2,\ k \in \mathbb{Z}$.
(c) M is a **Tucker** *type manifold, i.e. for any equivariant triangulation Λ of M and for any Tucker's labeling of vertices $V(\Lambda)$ there is a complementary edge.*
(d) M is a **Lyusternik-Shnirelman** *type manifold, i.e. for any cover $F_1, \ldots, F_{n+1}$ of M^n by $n+1$ closed (respectively, by $n+1$ open) sets, there is at least one set containing a pair $(x, T(x))$.*
(e) **(KKM type)** *For any covering of M by a family of $2n$ closed sets $\{C_1, C_{-1}, \ldots, C_n, C_{-n}\}$, where C_i and C_{-i} are antipodal, i. e. $C_{-i} = T(C_i)$, for all $i = 1, \ldots, d$, then there is k such that C_k and C_{-k} have a common intersection point.*
(f) **(Kakutani type)** *Let $F : M \to 2^{\mathbb{R}^n}$ be a set-valued function on M with a closed graph and the property that for all $x \in M$, $F(x) \neq \emptyset$, $F(x)$ is convex in $\mathbb{R}^n$ and there is $y \in F(x)$ such that $(-y) \in F(T(x))$. Then there is $x_0 \in M$ such that $F(x_0)$ covers the origin $0 \in \mathbb{R}^n$.*

Sperner's lemma is a statement about labellings (colorings) of triangulated simplices (d-balls). It is a discrete analog of the Brouwer fixed point theorem. In particular, we found a generalization of this lemma.

Theorem 2. *Let M^d be a compact oriented manifold with boundary. Let Λ be a triangulation of M with the vertex set $V(\Lambda)$. Then any labelling $L : V(\Lambda) \to \{1, 2, \ldots, d+1\}$ must contain at least $|\deg(L, \partial\Lambda)|$ fully-colored simplices.*

In this talk we also consider generalizations of the polytopal Sperner lemma, Tucker's lemma and the Ky Fan lemma for triangulations and quadrangulations.

References

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