A COMBINATORIAL MODEL OF THE LIPSCHITZ AND TEICHMÜLLER METRICS FOR SURFACES WITH PUNCTURES

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In the work [1] Ivan Dynnikov described a polynomial algorithm for the solution of the word problem in mapping class groups of punctured surfaces, where as the size of the algorithm's input he uses a modified version of the word length function of the mapping class group. Namely for a finite generating set $\mathcal{A}$ of the mapping class group of a punctured surface S Dynnikov defined *the zipped word length function* $\mathrm{zwl}_{\mathcal{A}}$ as follows:

$$\mathrm{zwl}_{\mathcal{A}}(\varphi) = \min_{\substack{\varphi = a_1^{k_1} \dots a_m^{k_m} \\ a_1,\dots,a_m \in \mathcal{A} \\ k_1,\dots,k_m \in \mathbb{Z}}} \sum_{i=1}^{m} \log_2(|k_i| + 1),$$

where $\varphi \in \mathrm{MCG}(S)$.

For special generating sets $\mathcal{A}$ he proved that the word problem is efficiently solvable with respect to $\mathrm{zwl}_{\mathcal{A}}$:

Theorem (Dynnikov). *Let S be a compact surface, $\mathcal{P} = (P_1, \dots, P_n) \in S$ a non-empty collection of pairwise distinct points such that the mapping class group $G = MCG(S \setminus \mathcal{P})$ is infinite. Let $\mathcal{A}$ be a finite generating set for G such that*

(1) *every element in $\mathcal{A}$ is a fractional power of a Dehn twist;*
(2) *every Dehn twist from G is conjugate to a fractional power of an element from $\mathcal{A}$*

Then the word problem in G is solvable in polynomial time with respect to $zwl_{\mathcal{A}}$.

The function $\mathrm{zwl}_{\mathcal{A}}$ determines the right-invariant metric $\rho_{\mathcal{A}}$ on $\mathrm{MCG}(S)$ as follows:

$$\rho_{\mathcal{A}}(\varphi, \psi) = \mathrm{zwl}_{\mathcal{A}}(\psi\varphi^{-1}),$$

where $\varphi, \psi \in \mathrm{MCG}(S)$.

It turns out that this metric is closely related to the asymmetric metric on the Teichmüller space described by Thurston in the work [2] and its symmetric version, called the Lipschitz metric(for the definition of this metric see e.g. [3]). In this talk we describe this relation and give the proof of the following theorem:

Theorem. *Let S be an oriented surface with non-empty set of punctures, ϵ a positive constant, σ a hyperbolic structure on S, lying in the ϵ-thick part of the Teichmüller space $\mathcal{T}_\epsilon(S)$, and $\mathcal{A}$ a finite generating set of $MCG(S)$ with the following properties:*

(1) *every element in $\mathcal{A}$ is a fractional power of a Dehn twist;*
(2) *every Dehn twist from G is conjugate to a fractional power of an element from $\mathcal{A}$*

Let also $i_\sigma \colon MCG(S) \to \mathcal{T}_\epsilon(S)$ be the map that sends $\varphi \in MCG(S)$ to the image of σ under φ. Then i_σ is a quasi-isometry from $MCG(S)$ equipped with the metric $\rho_{\mathcal{A}}$ to the thick part of $\mathcal{T}(S)$ equipped with the Lipschitz metric.

In the work [3] Young-Eun Choi and Kasra Rafi proved that the Lipschitz and the Teichmuller metrics are equal up to an additive constant in the thick part of the Teichmüller space. So from this result we obtain the following direct corollary of the previous theorem:

Corollary. *Let S be an oriented surface with non-empty set of punctures, ϵ a positive constant, σ a hyperbolic structure on S, lying in the ϵ-thick part of the Teichmüller space $\mathcal{T}_\epsilon(S)$, and $\mathcal{A}$ a finite generating set of $MCG(S)$ with the following properties:*

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(1) *every element in* $\mathcal{A}$ *is a fractional power of a Dehn twist;*
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Let also $i_\sigma\colon MCG(S) \to \mathcal{T}_\epsilon(S)$ *be the map that sends* $\varphi \in MCG(S)$ *to the image of* σ *under* φ*. Then* i_σ *is a quasi-isometry from* $MCG(S)$ *equipped with the metric* $\rho_{\mathcal{A}}$ *to the thick part of* $\mathcal{T}(S)$ *equipped with the Teichmüller metric.*

References

[1] I. Dynnikov. Counting intersections of normal curves, unpublished preprint
[2] W. P. Thurston, Minimal stretch maps between hyperbolic surfaces. Preprint 1986.
[3] Y.-E. Choi, K. Rafi, Comparison between Teichmüller and Lipschitz metrics, J. Lond. Math. Soc (2) 76 (2007), no. 3, 739-756.

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