BRACKETS IN THE FUNDAMENTAL GROUPS OF SURFACES AND IN THE PONTRYAGIN ALGEBRAS OF HIGH-DIMENSIONAL MANIFOLDS

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One of the most remarkable geometric features of a closed oriented surface is a natural symplectic structure on the moduli spaces of representations of its fundamental group, as shown by W. Goldman. These spaces consist of conjugacy classes of homomorphisms from the fundamental group, π, to appropriate Lie groups G. The symplectic structure on $Hom(\pi, G)/G$ incorporates the classical Kähler forms on the Teichmüller spaces ($G = \mathrm{PSL}(2,\mathbb{R})$), on the Jacobi varieties of Riemann surfaces ($G = \mathrm{U}(1)$), and on the Narasimhan–Seshadri moduli spaces of semistable vector bundles of rank $N \geq 1$ (where $G = \mathrm{U}(N)$). This also incorporates the Atiyah–Bott symplectic structure on $Hom(\pi, G)/G$ determined by a compact Lie group G and a non-degenerate $\mathrm{Ad}(G)$-invariant symmetric bilinear form on the Lie algebra of G. For a compact oriented surface with boundary, one obtains a weaker structure, namely, a Poisson bracket on $Hom(\pi, G)/G$, or, more precisely, a Poisson bracket in the algebra of conjugation-invariant smooth functions on the representation space $Hom(\pi, G)$. The latter bracket extends to a natural quasi-Poisson bracket in the algebra of all smooth functions on $Hom(\pi, G)$

In my lecture I will explain the algebraic structure in the fundamental group π of a surface which give rise to these Poisson brackets. The structure is a linear map

$$\mathbb{Z}[\pi] \otimes \mathbb{Z}[\pi] \to \mathbb{Z}[\pi] \otimes \mathbb{Z}[\pi]$$

defined in terms of intersections of loops on surfaces.

I will also discuss generalizations of these constructions to higher dimensions where the role of the (group ring of the) fundamental group is played by the so-called Pontryagin algebra. Namely, there is a bracket in representation algebras associated with the Pontryagin algebra of any smooth oriented manifold with non-empty boundary. For surfaces, our bracket is the quasi-Poisson bracket on the representation space $Hom(\pi, \mathrm{GL}_N)$ mentioned above. In higher dimensions, the representation algebras are graded, and the bracket in question is a Gerstenhaber bracket, i.e., it satisfies the axioms of a Poisson bracket with appropriate signs. This subject is closely related to the string topology of Chas-Sullivan.

This lecture is based on a series of joint papers with Gwénaël Massuyeau.

References

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