BRUNNIAN AND COHEN BRAIDS AND LIE ALGEBRAS

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We give a short survey of results about Brunnian and Cohen braids, their relations with Algebraic Topology, namely with homotopy groups of spheres and with the Lie algebras arising from descending central series.

Let M be a general connected surface, possibly with boundary components. We denote by $B_n(M)$ the n-strand braid group on a surface M. The operations

$$d_i\colon B_n(M) \to B_{n-1}(M)$$

are obtained by forgetting the i-th strand of a braid, $1 \leq i \leq n$. We study the system of equations

$$\begin{cases} d_1(\beta) = \alpha, \\ \dots \\ d_n(\beta) = \alpha. \end{cases} \tag{1}$$

where α is a braid in $B_{n-1}(M)$.

If α is the trivial braid, then the braid satisfying (1) is called *Brunnian*. Relations between Brunnian braids and homotopy groups of spheres were studied in particular in [1].

Apart from Brunnian braids the following example can be given of solutions of (1). Let α be the Garside element $\Delta_{n-1} \in B_{n-1}(M)$. Then $\Delta_n \in B_n(M)$ is a solution of system (1).

Theorem. *Let M be any connected 2-manifold such that $M \neq S^2$ or $\mathbb{R}P^2$ and let $\alpha \in B_{n-1}(M)$. Then the system of equations (1) for n-strand braids β has a solution if and only if α satisfies the condition that*

$$d_1\alpha = \ldots = d_{n-1}\alpha.$$

Let us define a set

$$\mathfrak{H}^B_n(M) = \{\beta \in B_n(M) \mid d_1\beta = d_2\beta = \cdots = d_n\beta\}.$$

Namely, $\mathfrak{H}^B_n(M)$ consists of n-strand braids such that it stays the same braid after removing any one of its strands. So, we call these braids *Cohen* braids.

Proposition. *Let M be any connected 2-manifold. Then the set $\mathfrak{H}^B_n(M)$ is a subgroup of $B_n(M)$. Moreover $d_i(\mathfrak{H}^B_n(M)) \subseteq \mathfrak{H}^B_{n-1}(M)$ and the map*

$$d_1 = d_2 = \cdots = d_n\colon \mathfrak{H}^B_n(M) \to \mathfrak{H}^B_{n-1}(M)$$

is a group homomorphism.

Proposition. *Let M be any connected 2-manifold. Let $n \geq 2$. Then $\mathfrak{H}^B_n(M) \cap P_n(M)$ is a subgroup of $\mathfrak{H}^B_n(M)$ of index 2.*

Let $\mathfrak{H}_n(M) = \mathfrak{H}^B_n(M) \cap P_n(M)$. Then $d_1(\mathfrak{H}_n(M)) \subseteq \mathfrak{H}_{n-1}(M)$.

Proposition. *Let M be any connected 2-manifold such that $M \neq S^2$ or $\mathbb{R}P^2$. Then there exists the following short exact sequence*

$$1 \to \mathrm{Brun}_n(M) \to \mathfrak{H}_n(M) \xrightarrow{d_1} \mathfrak{H}_{n-1}(M) \to 1 \tag{2}$$

which connects the nth, $(n-1)$th Cohen braid groups and Brunnian braids on n-strands.

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References

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