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ON THE CHIEF FACTORS OF PARABOLIC MAXIMAL SUBGROUPS IN FINITE SIMPLE GROUPS OF NORMAL LIE TYPE

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Abstract: For every finite group of normal Lie type, we specify the description of the parabolic factors of a parabolic maximal subgroup occurring in the unipotent radical. For every module, we determine the highest weight and dimension. The results are presented in the tables where these parameters are given.

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Introduction

One of the fundamental problems of group theory is the study of the subgroup structure of a given group. In the postclassification theory of finite groups, investigating their subgroups and representations has become an actual problem. The important class of permutation representation of finite groups of Lie type is given by its parabolic representations; i.e., the representations on the cosets modulo parabolic subgroups. In our previous works, we obtained a description for the primitive parabolic permutation representations of all groups of (normal and twisted) Lie type. This article continues the study of the properties of these representations; namely, the chief factors are studied of parabolic maximal subgroups in finite simple groups of normal Lie type.

Let G be a finite simple group of normal Lie type (a Chevalley group) over a finite field K of characteristic p and let $P = UL$ be a parabolic maximal subgroup in G, where U is the unipotent radical and L is the Levi complement in P. Suppose that $p \neq 2$ for the groups of type B_l, C_l, F_4 and $p > 3$ for the groups of type G_2. Then the results of [1] imply that the factors of the lower central series of U are chief factors of P and irreducible KL-modules. The number of these factors is independent of the field K but depends on the Lie type of G. In the exceptional cases, the commutator relations influencing the structure of unipotent subgroups behave in a special way and require special consideration.

In this article the author specifies the description of the factors of parabolic maximal subgroups P occurring in the unipotent radical under the above-indicated conditions. These chief factors are irreducible KL-modules. For every such module, we find the highest weight and dimension.

1. Notation and Auxiliary Results

The modular representations are an important tool in the study of the subgroup structure of finite groups. The irreducible representations of finite groups of Lie type can be parametrized by highest weights as it was done in the theory of representations of complex semisimple Lie algebras (for example, see [2–5]).

We will use the definitions and notation connected with the groups of Lie type from [6]. Let Φ be a root system for a group G, let π be the set of simple roots in Φ, and let Φ^+ be the corresponding set of positive roots. It is known that $G = \langle x_\beta(t) \mid t \in K, \beta \in \Phi \rangle$, and for each root $\beta \in \Phi$, the root subgroup $X_\beta = \{x_\beta(t) \mid t \in K\}$ is isomorphic to the field K. Given a G-module M and a character χ of a Cartan

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subgroup H in G, the nonzero space $M_\chi = \{v \in M \mid h \cdot v = \chi(h)v \text{ for all } h \in H\}$ is called a *weight space* and χ is called a *weight of the G-module M* (or the *weight of the corresponding representation*). A G-module M can be decomposed into a direct sum of weight spaces. There exists a one-dimensional subspace in M invariant under the action of a Borel subgroup in G. A vector v generating such a space is called a *maximal vector* for M. This is equivalent to the fact that $v \neq 0$ belongs to some weight space M_χ and is fixed under the action of all root subgroups X_β with positive roots β. For an irreducible G-module M, the weight χ is called its *highest weight*.

Given a subset J in π, denote by Φ_J the set of roots in Φ spanned by J and put $\Phi_J^+ = \Phi^+ \cap \Phi_J$. Let P be the standard parabolic subgroup in G corresponding to the root system Φ_J. A subgroup P is known to admit the Levi decomposition: $P = UL$, where $U = \langle X_\beta \mid \beta \in \Phi^+ \setminus \Phi_J^+ \rangle$ is the unipotent radical and $L = \langle H, X_\beta \mid \beta \in \Phi_J \rangle$ is the Levi complement in P. Denote the jth term of the lower central series of U by $U^{(j)}$, where $j \geq 1$ and $U = U^{(1)}$. Given $\beta \in \Phi^+$, write $\beta = \beta_J + \beta_{J'}$, where $\beta_J = \sum_{\alpha \in J} c_\alpha \alpha$ and $\beta_{J'} = \sum_{\alpha \in \pi \setminus J} d_\alpha \alpha$ $(0 \leq c_\alpha, d_\alpha \in \mathbb{Z})$. The number $ht(\beta) = \sum_{\alpha \in J} c_\alpha + \sum_{\alpha \in \pi \setminus J} d_\alpha$ is called the *height* of the root β. Following [1], call the number $\text{level}(\beta) = \sum_{\alpha \in \pi \setminus J} d_\alpha$ the *level* of the root β and refer to $\text{shape}(\beta) = \beta_{J'}$ as the *shape* of β. Given $j \geq 1$, put

$$U_j = \langle X_\beta \mid \beta \in \Phi^+, \text{level}(\beta) \geq j \rangle.$$

The Chevalley commutator formula [6, Theorem 5.2.2] implies that every subgroup U_j is normal in P and the quotient group U_j/U_{j+1} is isomorphic to $\prod X_\beta$, where the product is taken in some fixed order over all positive roots β for which $\text{level}(\beta) = j$. For all roots $\beta \in \Phi^+ \setminus \Phi_J^+$ with $\text{level}(\beta) = j$ and $\text{shape}(\beta) = S$, put

$$V_S = \Big(\prod_{\substack{\text{shape}(\beta)=S \\ \text{level}(\beta)=j}} X_\beta \Big) U_{j+1}/U_{j+1}.$$

Then write

$$U_j/U_{j+1} = \prod V_S,$$

where S ranges over the set of different shapes of level j in $\Phi^+ \setminus \Phi_J^+$.

For each $j \geq 1$, the group L acts by conjugation on U_j/U_{j+1}. If $\beta \in \Phi^+ \setminus \Phi_J$ and $\text{level}(\beta) = j$ then put $cx_\beta(t)U_{j+1} = x_\beta(ct)U_{j+1}$ for $c, t \in K$. Thus, U_j/U_{j+1} becomes a KL-module. The Chevalley commutator formula easily yields

Lemma 1. *If S is the shape of the roots of level j in $\Phi^+ \setminus \Phi_J^+$ then*

(1) *the vector space V_S over the field K is isomorphic to the external direct sum of root subgroups X_β for which* $\text{shape}(\beta) = S$ *and* $\text{level}(\beta) = j$*,*

(2) *V_S is normalized by L and is a KL-submodule in U_j/U_{j+1}.*

If A and B are normal subgroups in P, while B is a subgroup in A, and the quotient group A/B is a minimal normal subgroup in P/B, then A/B is called a *chief factor* of P. The following result is essential in finding the chief factors of a parabolic subgroup and composing the corresponding tables.

Lemma 2 [1, Theorem 2; 4, Theorem 17.6]. *Suppose that G is a finite simple group of normal Lie type over a finite field K of characteristic p, $p \neq 2$, for the groups of type B_l, C_l, F_4 and $p > 3$ for the groups of type G_2; while $P = UL$ is the parabolic subgroup in G corresponding to a root system Φ_J with unipotent radical U, and Levi complement L in P; the subgroups U_j and $U^{(j)}$ for $j \geq 1$ are as above. Then*

(1) *$U_j = U^{(j)}$ for each $j \geq 1$;*

(2) *for every shape S, among all roots $\beta \in \Phi^+ \setminus \Phi_J^+$ for which* $\text{shape}(\beta) = S$*, there exists a unique root β_S of maximal height, and the module V_S is an irreducible KL-module of highest weight β_S;*

(3) *if $|K| > 5$, S and S' are different shapes then V_S and $V_{S'}$ are nonequivalent KL-modules;*

(4) *for each* $j \geq 1$, *the module* U_j/U_{j+1} *is a direct sum of irreducible modules* V_S, *where the sum is taken in arbitrary order over all different shapes* S *of roots of level* j *in* $\Phi^+ \setminus \Phi^+_J$. *In particular,* U_j/U_{j+1} *is a completely reducible* KL*-module;*

(5) V_S *is a chief factor for* P.

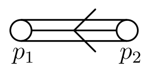
Put $\pi = \{p_1, \dots, p_l\}$, $k \in \{1, \dots, l\}$, $J = \Pi \setminus \{p_k\}$, $P_J = P_k$ and $\Phi_J = \Phi_k$. Then, for $\beta = \sum_{i=1}^{l} c_i p_i \in \Phi^+$ and $j \geq 1$, we have level$(\beta) = c_k$, shape$(\beta) = c_k p_k$, $U_j = \langle X_\beta \mid c_k \geq j \rangle = \prod_{c_k \geq j} X_\beta$, and

$$U_j/U_{j+1} = V_{jp_k} \cong \prod_{c_k = j} X_\beta.$$

2. The Exceptional Groups

We will provide the tables of chief factors for a parabolic maximal subgroup in an exceptional group of normal Lie type. In the first column of the table for each parabolic maximal subgroup P_k in G, we indicate all chief factors V_S occurring in its unipotent radical. The module V_S is isomorphic (in the additive notation) to a direct sum of root subgroups X_β, which are one-dimensional weight spaces of weight β. In the second column, we write down all such weights β. In the third column, we write down the highest weight β_S of the irreducible KL-module V_S; and, finally, in the last column, the dimension of V_S. Denote an arbitrary root $\sum_{i=1}^{l} c_i p_i$ in Φ by $c_1 c_2 \dots c_l$. The unit subgroup and the integer unity will be designated as 1. Let $K = GF(q)$, where q is a power of a prime p.

For the group $G_2(q)$ with the Dynkin diagram



the set of positive roots consists of the elements $10, 01, 11, 21, 31, 32$. Up to conjugacy, $G_2(q)$ has two parabolic maximal subgroups. Write down the Levi decomposition for each of them:

$$P_1 = \langle X_{10}, X_{11}, X_{21}, X_{31}, X_{32} \rangle \langle H, X_{\pm 01} \rangle,$$

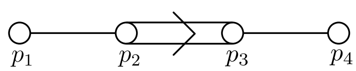
$$P_2 = \langle X_{01}, X_{11}, X_{21}, X_{31}, X_{32} \rangle \langle H, X_{\pm 10} \rangle.$$

The chief factors in P_1 are given by $V_{1p_1} = U_1/U_2 \cong X_{10}X_{11}$, $V_{2p_1} = U_2/U_3 \cong X_{21}$, and $V_{3p_1} = U_3/1 \cong X_{31}X_{32}$. Therefore, xU_2 is a maximal vector for the irreducible KL-module U_1/U_2 for any nonunit element $x \in X_{p_1+p_2}$. The element yU_3 is a maximal vector for the irreducible KL-module U_2/U_3 for every nonunit element $y \in X_{2p_1+p_2}$. Similarly, for P_2 we get $V_{1p_2} = U_1/U_2 \cong X_{01}X_{11}X_{21}X_{31}$, and $V_{2p_2} = U_2/1 \cong X_{32}$. Put the so-obtained results in Table 1.

Table 1. $G = G_2(q)$, $p > 3$.

$V_S \cong \bigoplus_\beta X_\beta$	β	β_S	$\dim V_S$
V_{1p_1}	$10, 11$	11	2
V_{2p_1}	21	21	1
V_{3p_1}	$31, 32$	32	2
V_{1p_2}	$01, 11, 21, 31$	31	4
V_{2p_2}	32	32	1

For the group $F_4(q)$ with the Dynkin diagram



the set of positive roots Φ^+ consists of the elements

$$1000, 0100, 0010, 0001, 1100, 0110, 0011, 1110, 0120, 0111, 1120, 1111,$$

$$0121, 1220, 1121, 0122, 1221, 1122, 1231, 1222, 1232, 1242, 1342, 2342.$$

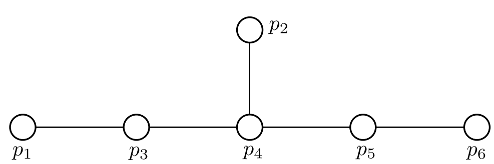
Up to conjugacy, $F_4(q)$ has four parabolic maximal subgroups $P_k = \langle H, X_\beta \mid \beta \in \Phi^+ \cup \Phi_k \rangle$, $1 \le k \le 4$. For each of these subgroups, put down in the table the chief factors, the weights of one-dimensional spaces contained in the corresponding irreducible module, the highest weight of the module, and its dimension (see Table 2).

Table 2. $G = F_4(q)$, $p \neq 2$.

V_S	Roots β with $V_S \cong \bigoplus_\beta X_\beta$	β_S	$\dim V_S$
V_{1p_1}	1000, 1100, 1110, 1120, 1111, 1220, 1121, 1221, 1122, 1231, 1222, 1232, 1242, 1342	1342	14
V_{2p_1}	2342	2342	1
V_{1p_2}	0100, 1100, 0110, 1110, 0120, 0111, 1120, 1111, 0121, 1121, 0122, 1122	1122	12
V_{2p_2}	1220, 1221, 1231, 1222, 1232, 1242	1242	6
V_{3p_2}	1342, 2342	2342	2
V_{1p_3}	0010, 0110, 0011, 1110, 0111, 1111	1111	6
V_{2p_3}	0120, 1120, 0121, 1220, 1121, 0122, 1221, 1122, 1222	1222	9
V_{3p_3}	1231, 1232	1232	2
V_{4p_3}	1242, 1342, 2342	2342	3
V_{1p_4}	0001, 0011, 0111, 1111, 0121, 1121, 1221, 1231	1231	8
V_{2p_4}	0122, 1122, 1222, 1232, 1242, 1342, 2342	2342	7

In each of the groups $E_l(q)$, where l is equal to 6, 7, or 8, there are l parabolic maximal subgroups $P_k = \langle H, X_\beta \mid \beta \in \Phi^+ \cup \Phi_k \rangle$, $1 \le k \le l$. For these groups, compose the tables similar to those for $G_2(q)$ and $F_4(q)$ (see Tables 3–5).

For $E_6(q)$ with the Dynkin diagram



the set of positive roots Φ^+ consists of the elements

$$100000, 010000, 001000, 000100, 000010, 000001, 101000, 010100, 001100,$$

$$000110, 000011, 101100, 011100, 010110, 001110, 000111, 111100, 101110,$$

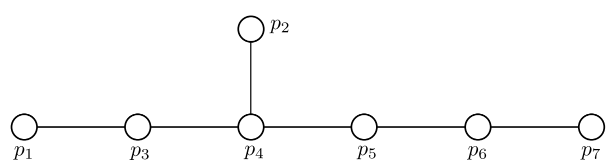
$$011110, 010111, 001111, 111110, 101111, 011210, 011111, 111210, 111111,$$

$$011211, 112210, 111211, 011221, 112211, 111221, 112221, 112321, 122321.$$

Table 3. $G = E_6(q)$.

V_S	Roots β with $V_S \cong \bigoplus_\beta X_\beta$	β_S	$\dim V_S$
V_{1p_1}	100000, 101000, 101100, 111100, 101110, 111110, 101111, 111210, 111111, 112210, 111211, 112211, 111221, 112221, 112321, 122321	122321	16
V_{1p_2}	010000, 010100, 011100, 010110, 111100, 011110, 010111, 111110, 011210, 011111, 111210, 111111, 011211, 112210, 111211, 011221, 112211, 111221, 112221, 112321	112321	20
V_{2p_2}	122321	122321	1
V_{1p_3}	001000, 101000, 001100, 101100, 011100, 001110, 111100, 101110, 011110, 001111, 111110, 101111, 011210, 011111, 111210, 111111, 011211, 111211, 011221, 111221	111221	20
V_{2p_3}	112210, 112211, 112221, 112321, 122321	122321	5
V_{1p_4}	000100, 010100, 001100, 000110, 101100, 011100, 010110, 001110, 000111, 111100, 101110, 011110, 010111, 001111, 111110, 101111, 011111, 111111	111111	18
V_{2p_4}	011210, 111210, 011211, 112210, 111211, 011221, 112211, 111221, 112221	112221	9
V_{3p_4}	112321, 122321	122321	2
V_{1p_5}	000010, 000110, 000011, 010110, 001110, 000111, 101110, 011110, 010111, 001111, 111110, 101111, 011210, 011111, 111210, 111111, 011211, 112210, 111211, 112211	112211	20
V_{2p_5}	011221, 111221, 112221, 112321, 122321	122321	5
V_{1p_6}	000001, 000011, 000111, 010111, 001111, 101111, 011111, 111111, 011211, 111211, 011221, 112211, 111221, 112221, 112321, 122321	122321	16

For $E_7(q)$ with the Dynkin diagram



the set of positive roots Φ^+ consists of the elements

1000000, 0100000, 0010000, 0001000, 0000100, 0000010, 0000001, 1010000, 0101000, 0011000, 0001100, 0000110, 0000011, 1011000, 0111000, 0101100, 0011100, 0001110, 0000111, 1111000, 1011100, 0111100, 0101110, 0011110, 0001111, 1111100, 1011110, 0112100, 0111110, 0101111, 0011111, 1112100, 1111110, 1011111, 0112110, 0111111, 1122100, 1112110, 1111111, 0112210, 0112111, 1122110, 1112210, 1112111, 0112211, 1122210, 1122111, 1112211, 0112221, 1123210, 1122211, 1112221, 1223210, 1123211, 1122221, 1223211, 1123221, 1223221, 1123321, 1223321, 1224321, 1234321, 2234321.

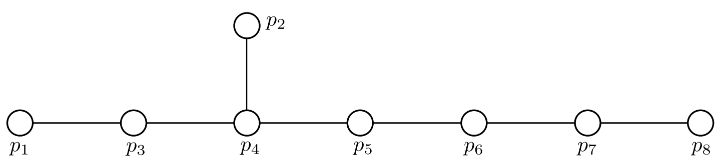
Table 4. $G = E_7(q)$.

V_S	Roots β with $V_S \cong \bigoplus_\beta X_\beta$	β_S	$\dim V_S$
V_{1p_1}	1000000, 1010000, 1011000, 1111000, 1011100, 1111100, 1011110, 1112100, 1111110, 1011111, 1122100, 1112110, 1111111, 1122110, 1112210, 1112111, 1122210, 1122111, 1112211, 1123210, 1122211, 1112221, 1223210, 1123211, 1122221, 1223211, 1123221, 1223221, 1123321, 1223321, 1224321, 1234321	1234321	32
V_{2p_1}	2234321	2234321	1
V_{1p_2}	0100000, 0101000, 0111000, 0101100, 1111000, 0111100, 0101110, 1111100, 0112100, 0111110, 0101111, 1112100, 1111110, 0112110, 0112210, 0112111, 1122110, 0111111, 1122100, 1112110, 1111111, 1122210, 1122111, 1112211, 0112221, 1112210, 1112111, 0112211, 1123210, 1122211, 1112221, 1123211, 1122221, 1123221, 1123321	1123321	35
V_{2p_2}	1223210, 1223211, 1223221, 1223321, 1224321, 1234321, 2234321	2234321	7
V_{1p_3}	0010000, 1010000, 0011000, 1011000, 0111000, 0011100, 1111000, 1011100, 0111100, 0011110, 1111100, 1011110, 0112100, 0111110, 0011111, 1112100, 1111110, 1011111, 0112110, 0111111, 1112110, 1111111, 0112210, 0112111, 1112210, 1112111, 0112211, 1112211, 0112221, 1112221	1112221	30
V_{2p_3}	1122100, 1122110, 1122210, 1122111, 1123210, 1122211, 1223210, 1123211, 1122221, 1223211, 1123221, 1223221, 1123321, 1223321, 1224321	1224321	15
V_{3p_3}	1234321, 2234321	2234321	2
V_{1p_4}	0001000, 0101000, 0011000, 0001100, 1011000, 0111000, 0101100, 0011100, 0001110, 1111000, 1011100, 0111100, 0101110, 0011110, 0001111, 1111100, 1011110, 0111110, 0101111, 0011111, 1111110, 1011111, 0111111, 1111111	1111111	24
V_{2p_4}	0112100, 1112100, 0112110, 1122100, 1112110, 0112210, 0112111, 1122110, 1112210, 1112111, 0112211, 1122210, 1122111, 1112211, 0112221, 1122211, 1112221, 1122221	1122221	18
V_{3p_4}	1123210, 1223210, 1123211, 1223211, 1123221, 1223221 1123321, 1223321	1223321	8
V_{4p_4}	1224321, 1234321, 2234321	2234321	3
V_{1p_5}	0000100, 0001100, 0000110, 0101100, 0011100, 0001110, 0000111, 1011100, 0111100, 0101110, 0011110, 0001111, 1111100, 1011110, 0112100, 0111110, 0101111, 0011111, 1112100, 1111110, 1011111, 0112110, 0111111, 1122100, 1112110, 1111111, 0112111, 1122110, 1112111, 1122111	1122111	30
V_{2p_5}	0112210, 1112210, 0112211, 1122210, 1112211, 0112221, 1123210, 1122211, 1112221, 1223210, 1123211, 1122221, 1223211, 1123221, 1223221	1223221	15

Table 4. $G = E_7(q)$ (continued).

V_S	Roots β with $V_S \cong \bigoplus_\beta X_\beta$	β_S	$\dim V_S$
V_{3p_5}	1123321, 1223321, 1224321, 1234321, 2234321	2234321	5
V_{1p_6}	0000010, 0000110, 0000011, 0001110, 0000111, 0101110, 0011110, 0001111, 1011110, 0111110, 0101111, 0011111, 1111110, 1011111, 0112110, 0111111, 1112110, 1111111, 0112210, 0112111, 1122110, 1112210, 1112111, 0112211, 1122210, 1122111, 1112211, 1123210, 1122211, 1223210, 1123211, 1223211	1223211	32
V_{2p_6}	0112221, 1112221, 1122221, 1123221, 1223221, 1123321, 1223321, 1224321, 1234321, 2234321	2234321	10
V_{1p_7}	0000001, 0000011, 0000111, 0001111, 0101111, 0011111, 1011111, 0111111, 1111111, 0112111, 1112111, 0112211, 1122111, 1112211, 0112221, 1122211, 1112221, 1123211, 1122221, 1223211, 1123221, 1223221, 1123321, 1223321, 1224321, 1234321, 2234321	2234321	27

For $E_8(q)$ with the Dynkin diagram



the set of positive roots Φ^+ of type E_8 consists of the elements

10000000, 01000000, 00100000, 00010000, 00001000, 00000100, 00000010, 00000001, 10100000, 01010000, 00110000, 00011000, 00001100, 00000110, 00000011, 10110000, 01110000, 01011000, 00111000, 00011100, 00001110, 00000111, 11110000, 10111000, 01111000, 01011100, 00111100, 00011110, 00001111, 11111000, 10111100, 01121000, 01111100, 01011110, 00111110, 00011111, 11121000, 11111100, 10111110, 01121100, 01111110, 01011111, 00111111, 11221000, 11121100, 11111110, 10111111, 01122100, 01121110, 01111111, 11221100, 11122100, 11121110, 11111111, 01122110, 01121111, 11222100, 11221110, 11122110, 11121111, 01122210, 01122111, 11232100, 11222110, 11221111, 11122210, 11122111, 01122211, 12232100, 11232110, 11222210, 11222111, 11122211, 01122221, 12232110, 11232210, 11232111, 11222211, 11122221, 12232210, 12232111, 11233210, 11232211, 11222221, 12233210, 12232211, 11233211, 11232221, 12243210, 12233211, 12232221, 11233221, 12343210, 12243211, 12233221, 11233321, 22343210, 12343211, 12243221, 12233321, 22343211, 12343221, 12243321, 22343221, 12343321, 12244321, 22343321, 12344321, 22344321, 12354321, 22354321, 13354321, 23354321, 22454321, 23454321, 23464321, 23465321, 23465421, 23465431, 23465432.

Table 5. $G = E_8(q)$.

V_S	Roots β with $V_S \cong \bigoplus_{\beta} X_\beta$	β_S	$\dim V_S$
V_{1p_1}	10000000, 10100000, 10110000, 11110000, 10111000, 11111000, 10111100, 11121000, 11111100, 10111110, 11221000, 11121100, 11111110, 10111111, 11221100, 11122100, 11121110, 11111111, 11222100, 11221110, 11122110, 11121111, 11232100, 11222110, 11221111, 11122210, 11122111, 12232100, 11232110, 11222210, 11222111, 11122211, 12232110, 11232210, 11232111, 11222211, 11122221, 12232210, 12232111, 11233210, 11232211, 11222221, 12233210, 12232211, 11233211, 11232221, 12243210, 12233211, 12232221, 11233221, 12343210, 12243211, 12233221, 11233321, 12343211, 12243221, 12233321, 12343221, 12243321, 12343321, 12244321, 12344321, 12354321, 13354321	13354321	64
V_{2p_1}	22343210, 22343211, 22343221, 22343321, 22344321, 22354321, 23354321, 22454321, 23454321, 23464321, 23465321, 23465421, 23465431, 23465432	23465432	14
V_{1p_2}	01000000, 01010000, 01110000, 01011000, 11110000, 01111000, 01011100, 11111000, 01121000, 01111100, 01011110, 11121000, 11111100, 01121100, 01111110, 01011111, 11221000, 11121100, 11111110, 01122100, 01121110, 01111111, 11221100, 11122100, 11121110, 11111111, 01122110, 01121111, 11222100, 11221110, 11122110, 11121111, 01122210, 01122111, 11232100, 11222110, 11221111, 11122210, 11122111, 01122211, 11232110, 11222210, 11222111, 11122211, 01122221, 11232210, 11232111, 11222211, 11122221, 11233210, 11232211, 11222221, 11233211, 11232221, 11233221, 11233321	11233321	56
V_{2p_2}	12232100, 12232110, 12232210, 12232111, 12233210, 12232211, 12243210, 12233211, 12232221, 12343210, 12243211, 12233221, 22343210, 12343211, 12243221, 12233321, 22343211, 12343221, 12243321, 22343221, 12343321, 12244321, 22343321, 12344321, 22344321, 12354321, 22354321, 22454321	22454321	28
V_{3p_2}	13354321, 23354321, 23454321, 23464321, 23465321, 23465421, 23465431, 23465432	23465432	8
V_{1p_3}	00100000, 10100000, 00110000, 10110000, 01110000, 00111000, 11110000, 10111000, 01111000, 00111100, 11111000, 10111100, 01121000, 01111100, 00111110, 11121000, 11111100, 10111110, 01121100, 01111110, 00111111, 11121100, 11111110, 10111111, 01122100, 01121110, 01111111, 11122100, 11121110, 11111111, 01122110, 01121111, 11122110, 11121111, 01122210, 01122111, 11122210, 11122111, 01122211, 11122211, 01122221, 11122221	11122221	42

Table 5. $G = E_8(q)$ (continued).

V_S	Roots β with $V_S \cong \bigoplus_\beta X_\beta$	β_S	$\dim V_S$
V_{2p_3}	11221000, 11221100, 11222100, 11221110, 11232100, 11222110, 11221111, 12232100, 11232110, 11222210, 11222111, 12232110, 11232210, 11232111, 11222211, 12232210, 12232111, 11233210, 11232211, 11222221, 12233210, 12232211, 11233211, 11232221, 12243210, 12233211, 12232221, 11233221, 12243211, 12233221, 11233321, 12243221, 12233321, 12243321, 12244321	12244321	35
V_{3p_3}	12343210, 22343210, 12343211, 22343211, 12343221, 22343221, 12343321, 22343321, 12344321, 22344321, 12354321, 22354321, 13354321, 23354321	23354321	14
V_{4p_3}	22454321, 23454321, 23464321, 23465321, 23465421, 23465431, 23465432	23465432	7
V_{1p_4}	00010000, 01010000, 00110000, 00011000, 10110000, 01110000, 01011000, 00111000, 00011100, 11110000, 10111000, 01111000, 01011100, 00111100, 00011110, 11111000, 10111100, 01111100, 01011110, 00111110, 00011111, 11111100, 10111110, 01111110, 01011111, 00111111, 11111110, 10111111, 01111111, 11111111	11111111	30
V_{2p_4}	01121000, 11121000, 01121100, 11221000, 11121100, 01122100, 01121110, 11221100, 11122100, 11121110, 01122110, 01121111, 11222100, 11221110, 11122110, 11121111, 01122210, 01122111, 11222110, 11221111, 11122210, 11122111, 01122211, 11222210, 11222111, 11122211, 01122221, 11222211, 11122221, 11222221	11222221	30
V_{3p_4}	11232100, 12232100, 11232110, 12232110, 11232210, 11232111, 12232210, 12232111, 11233210, 11232211, 12233210, 12232211, 11233211, 11232221, 12233211, 12232221, 11233221, 12233221, 11233321, 12233321	12233321	20
V_{4p_4}	12243210, 12343210, 12243211, 22343210, 12343211, 12243221, 22343211, 12343221, 12243321, 22343221, 12343321, 12244321, 22343321, 12344321, 22344321	22344321	15
V_{5p_4}	12354321, 22354321, 13354321, 23354321, 22454321, 23454321	23454321	6
V_{6p_4}	23464321, 23465321, 23465421, 23465431, 23465432	23465432	5
V_{1p_5}	00001000, 00011000, 00001100, 01011000, 00111000, 00011100, 00001110, 10111000, 01111000, 01011100, 00111100, 00011110, 00001111, 11111000, 10111100, 01121000, 01111100, 01011110, 00111110, 00011111, 11121000, 11111100, 10111110, 01121100, 01111110, 01011111, 00111111, 11221000, 11121100, 11111110, 10111111, 01121110, 01111111, 11221100, 11121110, 11111111, 01121111, 11221110, 11121111, 11221111	11221111	40

Table 5. $G = E_8(q)$ (continued).

V_S	Roots β with $V_S \cong \bigoplus_\beta X_\beta$	β_S	$\dim V_S$
V_{2p_5}	01122100, 11122100, 01122110, 11222100, 11122110, 01122210, 01122111, 11232100, 11222110, 11122210, 11122111, 01122211, 12232100, 11232110, 11222210, 11222111, 11122211, 01122221, 12232110, 11232210, 11232111, 11222211, 11122221, 12232210, 12232111, 11232211, 11222221, 12232211, 11232221, 12232221	12232221	30
V_{3p_5}	11233210, 12233210, 11233211, 12243210, 12233211, 11233221, 12343210, 12243211, 12233221, 11233321, 22343210, 12343211, 12243221, 12233321, 22343211, 12343221, 12243321, 22343221, 12343321, 22343321	22343321	20
V_{4p_5}	12244321, 12344321, 22344321, 12354321, 22354321, 13354321, 23354321, 22454321, 23454321, 23464321	23464321	10
V_{5p_5}	23465321, 23465421, 23465431, 23465432	23465432	4
V_{1p_6}	00000100, 00001100, 00000110, 00011100, 00001110, 00000111, 01011100, 00111100, 00011110, 00001111, 10111100, 01111100, 01011110, 00111110, 00011111, 11111100, 10111110, 01121100, 01111110, 01011111, 00111111, 11121100, 11111110, 10111111, 01122100, 01121110, 01111111, 11221100, 11122100, 11121110, 11111111, 01122110, 01121111, 11222100, 11221110, 11122110, 11121111, 01122111, 11232100, 11222110, 11221111, 11122111, 12232100, 11232110, 11222111, 12232110, 11232111, 12232111	12232111	48
V_{2p_6}	01122210, 11122210, 01122211, 11222210, 11122211, 01122221, 11232210, 11222211, 11122221, 12232210, 11233210, 11232211, 11222221, 12233210, 12232211, 11233211, 11232221, 12243210, 12233211, 12232221, 11233221, 12343210, 12243211, 12233221, 22343210, 12343211, 12243221, 22343211, 12343221, 22343221	22343221	30
V_{3p_6}	11233321, 12233321, 12243321, 12343321, 12244321, 22343321, 12344321, 22344321, 12354321, 22354321, 13354321, 23354321, 22454321, 23454321, 23464321, 23465321	23465321	16
V_{4p_6}	23465421, 23465431, 23465432	23465432	3
V_{1p_7}	00000010, 00000110, 00000011, 00001110, 00000111, 00011110, 00001111, 01011110, 00111110, 00011111, 10111110, 01111110, 01011111, 00111111, 11111110, 10111111, 01121110, 01111111, 11121110, 11111111, 01122110, 01121111, 11221110, 11122110, 11121111, 01122210, 01122111, 11222110, 11221111, 11122210, 11122111, 01122211, 11232110, 11222210, 11222111, 11122211, 12232110, 11232210, 11232111, 11222211, 12232210, 12232111, 11233210, 11232211, 12233210, 12232211, 11233211, 12243210, 12233211, 12343210, 12243211, 22343210, 12343211, 22343211	22343211	54

Table 5. $G = E_8(q)$ (continued).

V_S	Roots β with $V_S \cong \bigoplus_\beta X_\beta$	β_S	$\dim V_S$
V_{2p_7}	01122221, 11122221, 11222221, 11232221, 12232221, 11233221, 12233221, 11233321, 12243221, 12233321, 12343221, 12243321, 22343221, 12343321, 12244321, 22343321, 12344321, 22344321, 12354321, 22354321, 13354321, 23354321, 22454321, 23454321, 23464321, 23465321, 23465421	23465421	27
V_{3p_7}	23465431, 23465432	23465432	2
V_{1p_8}	00000001, 00000011, 00000111, 00001111, 00011111, 01011111, 00111111, 10111111, 01111111, 11111111, 01121111, 11121111, 01122111, 11221111, 11122111, 01122211, 11222111, 11122211, 01122221, 11232111, 11222211, 11122221, 12232111, 11232211, 11222221, 12232211, 11233211, 11232221, 12233211, 12232221, 11233221, 12243211, 12233221, 11233321, 12343211, 12243221, 12233321, 22343211, 12343221, 12243321, 22343221, 12343321, 12244321, 22343321, 12344321, 22344321, 12354321, 22354321, 13354321, 23354321, 22454321, 23454321, 23464321, 23465321, 23465421, 23465431	23465431	56
V_{2p_8}	23465432	23465432	1

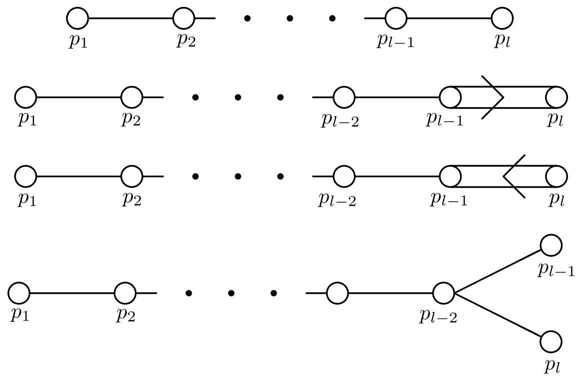
Table 6. Positive Roots for Classical Groups.

G	Φ^+
$A_l(q)$	$p_1, p_{12}, \dots, p_{1l}, p_2, p_{23}, \dots, p_{2l}, \dots, p_{l-1}, p_{l-1l}, p_l$
$B_l(q)$	$p_1, p_{12}, \dots, p_{1l}, p_2, p_{23}, \dots, p_{2l}, \dots, p_{l-1}, p_{l-1l}, p_l; p_1 + 2p_{2l}, p_{12} + 2p_{3l}, \dots, p_{1l-1} + 2p_l, p_2 + 2p_{3l}, p_{23} + 2p_{4l}, \dots, p_{2l-1} + 2p_l, \dots, p_{l-2} + 2p_{l-1l}, p_{l-2l-1} + 2p_l, p_{l-1} + 2p_l$
$C_l(q)$	$p_1, p_{12}, \dots, p_{1l}, p_2, p_{23}, \dots, p_{2l}, \dots, p_{l-1}, p_{l-1l}, p_l; p_1 + 2p_{2l-1} + p_l, p_{12} + 2p_{3l-1} + p_l, \dots, p_{1l-2} + 2p_{l-1} + p_l, p_2 + 2p_{3l-1} + p_l, p_{23} + 2p_{4l-1} + p_l, \dots, p_{2l-2} + 2p_{l-1} + p_l, \dots, p_{l-2} + 2p_{l-1} + p_l; 2p_{1l-1} + p_l, 2p_{2l-1} + p_l, \dots, 2p_{l-2l-1} + p_l, 2p_{l-1} + p_l$
$D_l(q)$	$p_1, p_{12}, \dots, p_{1l}, p_2, p_{23}, \dots, p_{2l}, \dots, p_{l-2}, p_{l-2l-1}, p_{l-2l}, p_{l-1}; p_1 + 2p_{2l-2} + p_{l-1l}, p_{12} + 2p_{3l-2} + p_{l-1l}, \dots, p_{1l-3} + 2p_{l-2} + p_{l-1l}, p_2 + 2p_{3l-2} + p_{l-1l}, p_{23} + 2p_{4l-2} + p_{l-1l}, \dots, p_{2l-3} + 2p_{l-2} + p_{l-1l}, \dots, p_{l-3} + 2p_{l-2} + p_{l-1l}; p_{1l-2} + p_l, p_{2l-2} + p_l, \dots, p_{l-3l-2} + p_l, p_{l-2} + p_l, p_l$

3. The Classical Groups

The classical groups of normal Lie are given by $A_l(q)$, $B_l(q)$, $C_l(q)$, and $D_l(q)$. For $1 \le i \le j \le l$ denote a positive root $p_i + p_{i+1} + \cdots + p_j$ by p_{ij} and, in particular, $p_i = p_{ii}$. In this notation, write down all positive roots for each classical group (see Table 6).

Proposition. *Suppose that $G \in \{A_l(q), B_l(q), C_l(q), D_l(q)\}$, $K = GF(q)$ and P is the parabolic* *maximal subgroup in G obtained by removing the kth vertex from the Dynkin diagram in the standard ordering of the vertices:*



Assume that q is odd for the groups of type B_l and C_l, while U is the unipotent radical of P, and the subgroups U_j and V_{jp_k}, $j \geq 1$, are as above. Then
(1) *If $G = A_l(q)$ then, for each $1 \leq k \leq l$,*
  (a) *$U = U_1$ is an abelian group;*
  (b) *$\dim V_{1p_k} = k(l-k+1)$.*
(2) *If $G = B_l(q)$ then*
  (a) *$U = U_1$ is an abelian group for $k = 1$;*
  (b) *the fragment of the chief series of P lying in U has the form $U = U_1 > U_2 > 1$ for $2 \leq k \leq l$;*
  (c) *$\dim V_{1p_k} = k(2l-2k+1)$ for $1 \leq k \leq l$;*
  (d) *$\dim V_{2p_k} = k(k-1)/2$ for $2 \leq k \leq l$.*
(3) *If $G = C_l(q)$ then*
  (a) *the fragment of the chief series of P lying in U has the form $U = U_1 > U_2 > 1$ for $1 \leq k \leq l-1$;*
  (b) *$U = U_1$ is an abelian group $k = l$;*
  (c) *$\dim V_{1p_k} = 2k(l-k)$ for $1 \leq k \leq l-1$ and $\dim V_{1p_l} = l(l+1)/2$;*
  (d) *$\dim V_{2p_k} = k(k+1)/2$ for $1 \leq k \leq l-1$.*
(4) *If $G = D_l(q)$ then*
  (a) *the fragment of the chief series of P lying in U has the form $U = U_1 > U_2 > 1$ for $2 \leq k \leq l-2$;*
  (b) *$U = U_1$ is an abelian group $k \in \{1, l-1, l\}$;*
  (C) *$\dim V_{1p_k} = 2k(l-k)$ for $1 \leq k \leq l-2$ and $\dim V_{1p_k} = l(l-1)/2$ for $k \in \{l-1, l\}$;*
  (d) *$\dim V_{2p_k} = k(k-1)/2$ for $2 \leq k \leq l-2$.*

Proof. Case $G = A_l(q)$. If $1 \leq k \leq l$ then

$$\Phi^+ \setminus \Phi_k^+ = \{p_{1k}, p_{1k+1}, \dots, p_{1l}, p_{2k}, p_{2k+1}, \dots, p_{2l}, \dots, p_{k-1k}, p_{k-1k+1}, \dots, p_{k-1l}, p_k, p_{kk+1}, p_{kk+2}, \dots, p_{kl}\}.$$

For $\beta \in \Phi^+ \setminus \Phi_k^+$, we have shape$(\beta) = 1p_k$ and $U = U_1$. Therefore, U is an abelian group and there exists only one chief factor $V_{1p_k} = U_1/1$ of P included in U. For every $t \in K$, the element $x_{p_{1l}}(t) \in U$ is a maximal vector and the root p_{1l} is the highest weight of the KL-module V_{1p_k}. Since $\dim V_{1p_k} = |\Phi^+ \setminus \Phi_k^+| = k(l-k+1)$, Case (1) of the proposition is proved.

Case $G = B_l(q)$. If $1 \leq k \leq l$ then

$$\Phi^+ \setminus \Phi_k^+ = \{p_{1k}, p_{1k+1}, \dots, p_{1l}, p_{2k}, p_{2k+1}, \dots, p_{2l}, \dots, p_{k-1k}, p_{k-1k+1}, \dots, p_{k-1l}, p_k, p_{kk+1}, \dots, p_{kl}; p_1 + 2p_{2l}, p_{12} + 2p_{3l}, \dots, p_{1l-1} + 2p_l, p_2 + 2p_{3l}, p_{23} + 2p_{4l}, \dots, p_{2l-1} + 2p_l, \dots, p_{k-1} + 2p_{kl}, p_{k-1k} + 2p_{k+1l}, \dots, p_{k-1l-1} + 2p_l, p_k + 2p_{k+1l}, p_{kk+1} + 2p_{k+2l}, \dots, p_{kl-1} + 2p_l\};$$

in particular, $\Phi^+ \setminus \Phi_1^+ = \{p_1, p_{12}, \ldots, p_{1l}; p_1 + 2p_{2l}, p_{12} + 2p_{3l}, \ldots, p_{1l-1} + 2p_l\}$. For $\beta \in \Phi^+ \setminus \Phi_1^+$, we have shape$(\beta) = 1p_1$, $U = U_1$ is an abelian group, there exists only one chief factor $V_{1p_1} = U_1/1$ for P included in U, and $\dim V_{1p_1} = 2l - 1$. For every $t \in K$, the element $x_{p_1+2p_{2l}}(t) \in U$ is a maximal vector and the root $p_1 + 2p_{2l}$ is the highest weight of the KL-module V_{1p_1}.

If $\beta \in \Phi^+ \setminus \Phi_k^+$ and $2 \le k \le l-1$ then shape$(\beta) = 1p_k$ or shape$(\beta) = 2p_k$. We have the series $U = U_1 > U_2 > 1$ and two chief factors $V_{1p_k} = U_1/U_2$ and $V_{2p_k} = U_2/1$ of P. The root $p_{1k} + 2p_{k+1l}$ is the highest weight of the KL-module V_{1p_k} and $p_1 + 2p_{2l}$ is the highest weight of V_{2p_k}.

If $k = l$ then

$$\Phi^+ \setminus \Phi_l^+ = \{p_{1l}, p_{2l}, \ldots, p_{l-1l}, p_l; p_1 + 2p_{2l}, p_{12} + 2p_{3l}, \ldots, p_{1l-1} + 2p_l, p_2 + 2p_{3l}, p_{23} + 2p_{4l}, \ldots, p_{2l-1} + 2p_l, \ldots, p_{l-1} + 2p_l\}.$$

The roots $p_{1l}, p_{2l}, \ldots, p_{l-1l}, p_l$ have shape $1p_l$, and the remaining roots in $\Phi^+ \setminus \Phi_l^+$ have shape $2p_l$; therefore, we obtain a fragment $U = U_1 > U_2 > 1$ of the central series of P with two factors $V_{1p_l} = U_1/U_2$ and $V_{2p_l} = U_2/1$. The root p_{1l} is the highest weight of the first KL-module, whereas $p_1 + 2p_{2l}$ is the highest weight of the second. Now, it is easy to compute the dimensions of V_{1p_k} and V_{2p_k}.

Case $G = C_l(q)$. If $1 \le k \le l-2$ then

$$\begin{aligned}\Phi^+ \setminus \Phi_k^+ = \{&p_{1k}, p_{1k+1}, \ldots, p_{1l}, p_{2k}, p_{2k+1}, \ldots, p_{2l}, \ldots, p_{k-1k}, p_{k-1k+1}, \ldots, \\ &p_{k-1l}, p_k, p_{kk+1}, \ldots, p_{kl}; p_1 + 2p_{2l-1} + p_l, p_{12} + 2p_{3l-1} + p_l, \\ &\ldots, p_{1l-2} + 2p_{l-1} + p_l, p_2 + 2p_{3l-1} + p_l, p_{23} + 2p_{4l-1} + p_l, \ldots, \\ &p_{2l-2} + 2p_{l-1} + p_l, \ldots, p_{k-1} + 2p_{kl-1} + p_l, p_{k-1k} + 2p_{k+1l-1} + p_l, \\ &\ldots, p_{k-1l-2} + 2p_{l-1} + p_l, p_k + 2p_{k+1l-1} + p_l, p_{kk+1} + 2p_{k+2l-1} + p_l, \\ &\ldots, p_{kl-2} + 2p_{l-1} + p_l; 2p_{1l-1} + p_l, 2p_{2l-1} + p_l, \ldots, 2p_{kl-1} + p_l\},\end{aligned}$$

$$\begin{aligned}\Phi^+ \setminus \Phi_{l-1}^+ = \{&p_{1l-1}, p_{1l}, p_{2l-1}, p_{2l}, \ldots, p_{l-2l-1}, p_{l-2l}, p_{l-1}, p_{l-1l}; p_1 + 2p_{2l-1} + p_l, \\ &p_{12} + 2p_{3l-1} + p_l, \ldots, p_{1l-2} + 2p_{l-1} + p_l, p_2 + 2p_{3l-1} + p_l, \\ &p_{23} + 2p_{4l-1} + p_l, \ldots, p_{2l-2} + 2p_{l-1} + p_l, \ldots, p_{l-2} + 2p_{l-1} + p_l; \\ &2p_{1l-1} + p_l, 2p_{2l-1} + p_l, \ldots, 2p_{l-2l-1} + p_l, 2p_{l-1} + p_l\},\end{aligned}$$

$$\begin{aligned}\Phi^+ \setminus \Phi_l^+ = \{&p_{1l}, p_{2l}, \ldots, p_{l-1l}, p_l; p_1 + 2p_{2l-1} + p_l, p_{12} + 2p_{3l-1} + p_l, \ldots, \\ &p_{1l-2} + 2p_{l-1} + p_l, p_2 + 2p_{3l-1} + p_l, \ldots, p_{2l-2} + 2p_{l-1} + p_l, \ldots, \\ &p_{l-3} + 2p_{l-2l-1} + p_l, p_{l-3l-2} + 2p_{l-1} + p_l, p_{l-2} + 2p_{l-1} + p_l; \\ &2p_{1l-1} + p_l, 2p_{2l-1} + p_l, \ldots, 2p_{l-2l-1} + p_l, 2p_{l-1} + p_l\}.\end{aligned}$$

If $\beta \in \Phi^+ \setminus \Phi_k^+$ and $1 \le k \le l-2$ then shape$(\beta) = 1p_k$ or shape$(\beta) = 2p_k$. We have the series $U = U_1 > U_2 > 1$ and two chief factors V_{1p_k} and V_{2p_k} of P. The root $p_{1k} + 2p_{k+1l-1} + p_l$ is the highest weight of the first KL-module V_{1p_k}, whereas $2p_{1l-1} + p_l$ is the highest weight of the second.

For $k = l-1$ the situation is similar excluding the highest weight of $V_{1p_{l-1}}$.

If $\beta \in \Phi^+ \setminus \Phi_l^+$ then shape$(\beta) = 1p_l$, and U is an abelian group. For every $t \in K$, the element $x_{2p_{1l-1}+p_l}(t) \in U$ is a maximal vector and the root $2p_{1l-1} + p_l$ is the highest weight of V_{1p_l}.

Calculate the number of roots for each shape and thus find the dimensions of V_{1p_k} and V_{2p_k}, $1 \le k \le l$.

Case $G = D_l(q)$. Given a root β, from

$$\Phi^+ \setminus \Phi_1^+ = \{p_1, p_{12}, \ldots, p_{1l}; p_{1l-2} + p_l; p_1 + 2p_{2l-2} + p_{l-1l}, p_{12} + 2p_{3l-2} + p_{l-1l}, \ldots, p_{1l-3} + 2p_{l-2} + p_{l-1l}\}$$

Table 7.

V_S for $A_l(q)$	Roots β with $V_S \cong \bigoplus_\beta X_\beta$
V_{1p_k} k is arbitrary	$p_{1k}, p_{1k+1}, \dots, p_{1l}, p_{2k}, p_{2k+1}, \dots, p_{2l}, \dots, p_{k-1k}, p_{k-1k+1}, \dots,$ $p_{k-1l}, p_k, p_{kk+1}, \dots, p_{kl}$

V_S for $B_l(q)$	β
V_{1p_1}	$p_1, p_{12}, \dots, p_{1l}; p_1 + 2p_{2l}, p_{12} + 2p_{3l}, \dots, p_{1l-1} + 2p_l$
V_{1p_k} $2 \le k \le l-1$	$p_{1k}, p_{1k+1}, \dots, p_{1l}, p_{2k}, p_{2k+1}, \dots, p_{2l}, \dots, p_k, p_{kk+1}, \dots, p_{kl};$ $p_{1k} + 2p_{k+1l}, p_{1k+1} + 2p_{k+2l}, \dots, p_{1l-1} + 2p_l, p_{2k} + 2p_{k+1l},$ $p_{2k+1} + 2p_{k+2l}, \dots, p_{2l-1} + 2p_l, \dots, p_k + 2p_{k+1l},$ $p_{kk+1} + 2p_{k+2l}, \dots, p_{kl-1} + 2p_l$
V_{2p_k} $2 \le k \le l-1$	$p_1 + 2p_{2l}, p_{12} + 2p_{3l}, \dots, p_{1k-1} + 2p_{kl}, p_2 + 2p_{3l},$ $p_{23} + 2p_{4l}, \dots, p_{2k-1} + 2p_{kl}, \dots, p_{k-2} + 2p_{k-1l},$ $p_{k-2k-1} + 2p_{kl}, p_{k-1} + 2p_{kl}$
V_{1p_l}	$p_{1l}, p_{2l}, \dots, p_{l-1l}, p_l$
V_{2p_l}	$p_1 + 2p_{2l}, p_{12} + 2p_{3l}, \dots, p_{1l-1} + 2p_l,$ $p_2 + 2p_{3l}, p_{23} + 2p_{4l}, \dots, p_{2l-1} + 2p_l, \dots, p_{l-1} + 2p_l$

V_S for $C_l(q)$	β
V_{1p_k} $1 \le k \le l-2$	$p_{1k}, p_{1k+1}, \dots, p_{1l}, p_{2k}, p_{2k+1}, \dots, p_{2l}, \dots, p_{k-1k}, p_{k-1k+1},$ $\dots, p_{k-1l}, p_k, p_{kk+1}, \dots, p_{kl}; p_{1k} + 2p_{k+1l-1} + p_l,$ $p_{1k+1} + 2p_{k+2l-1} + p_l, \dots, p_{1l-2} + 2p_{l-1} + p_l,$ $p_{2k} + 2p_{k+1l-1} + p_l, \dots, p_{2l-2} + 2p_{l-1} + p_l,$ $\dots, p_{k-1k} + 2p_{k+1l-1} + p_l, p_{k-1k+1} + 2p_{k+2l-1} + p_l, \dots,$ $p_{k-1l-2} + 2p_{l-1} + p_l, p_k + 2p_{k+1l-1} + p_l,$ $p_{kk+1} + 2p_{k+2l-1} + p_l, \dots, p_{kl-2} + 2p_{l-1} + p_l$
V_{2p_k} $1 \le k \le l-2$	$p_1 + 2p_{2l-1} + p_l, p_{12} + 2p_{3l-1} + p_l, \dots, p_{1k-1} + 2p_{kl-1} + p_l,$ $p_2 + 2p_{3l-1} + p_l, \dots, p_{2k-1} + 2p_{kl-1} + p_l, \dots,$ $p_{k-1} + 2p_{kl-1} + p_l; 2p_{1l-1} + p_l, 2p_{2l-1} + p_l, \dots, 2p_{kl-1} + p_l$
$V_{1p_{l-1}}$	$p_{1l-1}, p_{1l}, p_{2l-1}, p_{2l}, \dots, p_{l-2l-1}, p_{l-2l}, p_{l-1}, p_{l-1l}$
$V_{2p_{l-1}}$	$p_1 + 2p_{2l-1} + p_l, p_{12} + 2p_{3l-1} + p_l, \dots, p_{1l-2} + 2p_{l-1} + p_l,$ $p_2 + 2p_{3l-1} + p_l, p_{23} + 2p_{4l-1} + p_l, \dots, p_{2l-2} + 2p_{l-1} + p_l,$ $\dots, p_{l-2} + 2p_{l-1} + p_l;$ $2p_{1l-1} + p_l, 2p_{2l-1} + p_l, \dots, 2p_{l-2l-1} + p_l, 2p_{l-1} + p_l$
V_{1p_l}	$p_{1l}, p_{2l}, \dots, p_{l-1l}, p_l; p_1 + 2p_{2l-1} + p_l, p_{12} + 2p_{3l-1} + p_l, \dots,$ $p_{1l-2} + 2p_{l-1} + p_l, p_2 + 2p_{3l-1} + p_l, p_{23} + 2p_{4l-1} + p_l, \dots,$ $p_{2l-2} + 2p_{l-1} + p_l, \dots, p_{l-2} + 2p_{l-1} + p_l;$ $2p_{1l-1} + p_l, 2p_{2l-1} + p_l, \dots, 2p_{l-2l-1} + p_l, 2p_{l-1} + p_l$

we have shape$(\beta) = 1p_1$; hence, U is an abelian group. For every $t \in K$, the element $x_{p_1+2p_{2l-2}+p_{l-1l}}(t) \in U$ is a maximal vector and the root $p_1 + 2p_{2l-2} + p_{l-1l}$ is the highest weight of the KL-module V_{1p_1}. Since $\dim V_{1p_1} = |\Phi^+ \setminus \Phi_1^+|$, we have $\dim V_{1p_k} = 2l - 2$.

Table 7 (continued).

V_S for $D_l(q)$	β
V_{1p_1}	$p_1, p_{12}, \dots, p_{1l}; p_{1l-2}+p_l, p_1+2p_{2l-2}+p_{l-1l},$ $p_{12}+2p_{3l-2}+p_{l-1l}, \dots, p_{1l-3}+2p_{l-2}+p_{l-1l}$
V_{1p_k} $2 \le k \le l-3$	$p_{1k}, p_{1k+1}, \dots, p_{1l}, p_{2k}, p_{2k+1}, \dots, p_{2l} \dots, p_{k-1k}, p_{k-1k+1}, \dots,$ $p_{k-1l}, p_k, p_{kk+1}, \dots, p_{kl}; p_{1l-2}+p_l, p_{2l-2}+p_l, \dots, p_{kl-2}+p_l;$ $p_{1k}+2p_{k+1l-2}+p_{l-1l}, p_{1k+1}+2p_{k+2l-2}+p_{l-1l}, \dots,$ $p_{1l-3}+2p_{l-2}+p_{l-1l}, p_{2k}+2p_{k+1l-2}+p_{l-1l},$ $p_{2k+1}+2p_{k+2l-2}+p_{l-1l}, \dots, p_{2l-3}+2p_{l-2}+p_{l-1l}, \dots,$ $p_{k-1k}+2p_{k+1l-2}+p_{l-1l}, p_{k-1k+1}+2p_{k+2l-2}+p_{l-1l}, \dots,$ $p_{k-1l-3}+2p_{l-2}+p_{l-1l}, p_k+2p_{k+1l-2}+p_{l-1l},$ $p_{kk+1}+2p_{k+2l-2}+p_{l-1l}, \dots, p_{kl-3}+2p_{l-2}+p_{l-1l}$
V_{2p_k} $2 \le k \le l-3$	$p_1+2p_{2l-2}+p_{l-1l}, p_{12}+2p_{3l-2}+p_{l-1l}, \dots,$ $p_{1k-1}+2p_{kl-2}+p_{l-1l}, p_2+2p_{3l-2}+p_{l-1l}, \dots,$ $p_{2k-1}+2p_{kl-2}+p_{l-1l}, \dots, p_{k-1}+2p_{kl-2}+p_{l-1l}$
$V_{1p_{l-2}}$	$p_{1l-2}, p_{1l-1}, p_{1l}, p_{2l-2}, p_{2l-1}, p_{2l}, \dots, p_{l-2}, p_{l-2l-1}, p_{l-2l};$ $p_{1l-2}+p_l, p_{2l-2}+p_l, \dots, p_{l-2}+p_l$
$V_{2p_{l-2}}$	$p_1+2p_{2l-2}+p_{l-1l}, p_{12}+2p_{3l-2}+p_{l-1l}, \dots,$ $p_{1l-3}+2p_{l-2}+p_{l-1l}, p_2+2p_{3l-2}+p_{l-1l}, p_{23}+2p_{4l-2}+p_{l-1l},$ $\dots, p_{2l-3}+2p_{l-2}+p_{l-1l}, \dots, p_{l-4}+2p_{l-3l-2}+p_{l-1l},$ $p_{l-4,l-3}+2p_{l-2}+p_{l-1l}, p_{l-3}+2p_{l-2}+p_{l-1l}$
$V_{1p_{l-1}}$	$p_{1l-1}, p_{1l}, p_{2l-1}, p_{2l}, \dots, p_{l-2l-1}, p_{l-2l}, p_{l-1};$ $p_1+2p_{2l-2}+p_{l-1l}, p_{12}+2p_{3l-2}+p_{l-1l}, \dots,$ $p_{1l-3}+2p_{l-2}+p_{l-1l}, p_2+2p_{3l-2}+p_{l-1l}, \dots,$ $p_{2l-3}+2p_{l-2}+p_{l-1l}, \dots, p_{l-3}+2p_{l-2}+p_{l-1l}$
V_{1p_l}	$p_{1l}, p_{2l}, \dots, p_{l-2l}; p_1+2p_{2l-2}+p_{l-1l}, p_{12}+2p_{3l-2}+p_{l-1l},$ $\dots, p_{1l-3}+2p_{l-2}+p_{l-1l}, p_2+2p_{3l-2}+p_{l-1l},$ $p_{23}+2p_{4l-2}+p_{l-1l}, \dots, p_{2l-3}+2p_{l-2}+p_{l-1l},$ $\dots, p_{l-3}+2p_{l-2}+p_{l-1l};$ $p_{1l-2}+p_l, p_{2l-2}+p_l, \dots, p_{l-3l-2}+p_l, p_{l-2}+p_l, p_l$

For $2 \le k \le l-3$, we infer

$$\begin{aligned}\Phi^+ \setminus \Phi_k^+ = \{&p_{1k}, p_{1k+1}, \dots, p_{1l}, p_{2k}, p_{2k+1}, \dots, p_{2l}, \dots, p_k, p_{kk+1}, \dots, p_{kl};\\ &p_{1l-2}+p_l, p_{2l-2}+p_l, \dots, p_{kl-2}+p_l; p_1+2p_{2l-2}+p_{l-1l},\\ &p_{12}+2p_{3l-2}+p_{l-1l}, \dots, p_{1l-3}+2p_{l-2}+p_{l-1l}, p_2+2p_{3l-2}+p_{l-1l},\\ &\dots, p_{2l-3}+2p_{l-2}+p_{l-1l}, \dots, p_{k-1}+2p_{kl-2}+p_{l-1l},\\ &p_{k-1k}+2p_{k+1l-2}+p_{l-1l}, \dots, p_{k-1l-3}+2p_{l-2}+p_{l-1l},\\ &p_k+2p_{k+1l-2}+p_{l-1l}, \dots, p_{kl-3}+2p_{l-2}+p_{l-1l}\}.\end{aligned}$$

The roots

$$\begin{aligned}&p_{1k}, p_{1k+1}, \dots, p_{1l}, p_{2k}, p_{2k+1}, \dots, p_{2l}, \dots, p_k, p_{kk+1}, \dots, p_{kl}; p_{1l-2}+p_l,\\ &p_{2l-2}+p_l, \dots, p_{kl-2}+p_l; p_{1k}+2p_{k+1l-2}+p_{l-1l}, p_{1k+1}+2p_{k+2l-2}+p_{l-1l}, \dots,\\ &p_{1l-3}+2p_{l-2}+p_{l-1l}, p_{2k}+2p_{k+1l-2}+p_{l-1l}, p_{2k+1}+2p_{k+2l-2}+p_{l-1l}, \dots,\\ &p_{2l-3}+2p_{l-2}+p_{l-1l}, \dots, p_{k-1k}+2p_{k+1l-2}+p_{l-1l}, p_{k-1k+1}+2p_{k+2l-2}+p_{l-1l},\\ &\dots, p_{k-1l-3}+2p_{l-2}+p_{l-1l}, p_k+2p_{k+1l-2}+p_{l-1l}, \dots, p_{kl-3}+2p_{l-2}+p_{l-1l}\end{aligned}$$

Table 8. The Chief Factors, the Highest Weights, and the Powers for Classical Groups.

G	k	V_S	β_S	$\dim V_S$
$A_l(q)$	is arbitrary	V_{1p_k}	p_{1l}	$k(l-k+1)$
$B_l(q)$, $p \neq 2$	$1 \leq k \leq l-1$	V_{1p_k}	$p_{1k}+2p_{k+1l}$	$k(2l-2k+1)$
$B_l(q)$, $p \neq 2$	$2 \leq k \leq l$	V_{2p_k}	p_1+2p_{2l}	$k(k-1)/2$
$B_l(q)$, $p \neq 2$	l	V_{1p_l}	p_{1l}	l
$C_l(q)$, $p \neq 2$	$1 \leq k \leq l-2$	V_{1p_k}	$p_{1k}+2p_{k+1l-1}+p_l$	$2k(l-k)$
$C_l(q)$, $p \neq 2$	$1 \leq k \leq l-2$	V_{2p_k}	$2p_{1l-1}+p_l$	$k(k+1)/2$
$C_l(q)$, $p \neq 2$	$l-1$	$V_{1p_{l-1}}$	p_{1l}	$2(l-1)$
$C_l(q)$, $p \neq 2$	$l-1$	$V_{2p_{l-1}}$	$2p_{1l-1}+p_l$	$l(l-1)/2$
$C_l(q)$, $p \neq 2$	l	V_{1p_l}	$2p_{1l-1}+p_l$	$l(l+1)/2$
$D_l(q)$	$1 \leq k \leq l-3$	V_{1p_k}	$p_{1k}+2p_{k+1l-2}+p_{l-1l}$	$2k(l-k)$
$D_l(q)$	$l-2$	$V_{1p_{l-2}}$	p_{1l}	$4(l-2)$
$D_l(q)$	$2 \leq k \leq l-2$	V_{2p_k}	$p_1+2p_{2l-2}+p_{l-1l}$	$k(k-1)/2$
$D_l(q)$	$l-1$	$V_{1p_{l-1}}$	$p_1+2p_{2l-2}+p_{l-1l}$	$l(l-1)/2$
$D_l(q)$	l	V_{1p_l}	$p_1+2p_{2l-2}+p_{l-1l}$	$l(l-1)/2$

have shape $1p_k$, whereas the remaining roots in $\Phi^+ \setminus \Phi_k^+$ have shape $2p_k$; therefore, for $2 \leq k \leq l-3$ we get the chief series $U = U_1 > U_2 > 1$. The root $p_{1k}+2p_{k+1l-2}+p_{l-1l}$ is the highest weight of the KL-module $V_{1p_k} = U_1/U_2$, and $p_1+2p_{2l-2}+p_{l-1l}$ is the highest weight of $V_{2p_k} = U_2/1$. Next,

$$\begin{gathered}\Phi^+ \setminus \Phi_{l-2}^+ = \{p_{1l-2}, p_{1l-1}, p_{1l}, p_{2l-2}, p_{2l-1}, p_{2l}, \dots, p_{l-2}, p_{l-2l-1}, p_{l-2l}; \\ p_1+2p_{2l-2}+p_{l-1l}, p_{12}+2p_{3l-2}+p_{l-1l}, \dots, p_{1l-3}+2p_{l-2}+p_{l-1l}, \\ p_2+2p_{3l-2}+p_{l-1l}, p_{23}+2p_{4l-2}+p_{l-1l}, \dots, p_{2l-3}+2p_{l-2}+p_{l-1l}, \\ \dots, p_{l-4}+2p_{l-3l-2}+p_{l-1l}, p_{l-4l-3}+2p_{l-2}+p_{l-1l}, \\ p_{l-3}+2p_{l-2}+p_{l-1l}; p_{1l-2}+p_l, p_{2l-2}+p_l, \dots, p_{l-2}+p_l\}.\end{gathered}$$

The roots $p_{1l-2}, p_{1l-1}, p_{1l}, p_{2l-2}, p_{2l-1}, p_{2l}, \dots, p_{l-2}, p_{l-2l-1}, p_{l-2l}; p_{1l-2}+p_l, p_{2l-2}+p_l, \dots, p_{l-2}+p_l$ have shape $1p_{l-2}$, and the remaining roots in $\Phi^+ \setminus \Phi_{l-2}^+$ have shape $2p_{l-2}$; therefore, for $k = l-2$, we have the series $U = U_1 > U_2 > 1$, and two chief factors $V_{1p_{l-2}} = U_1/U_2$ and $V_{2p_{l-2}} = U_2/1$ of P. The root p_{1l} is the highest weight of the KL-module $V_{1p_{l-2}}$, and $p_1+2p_{2l-2}+p_{l-1l}$ is the highest wight of $V_{2p_{l-2}}$. Consider $k \in \{l-1, l\}$. We infer

$$\begin{gathered}\Phi^+ \setminus \Phi_{l-1}^+ = \{p_{1l-1}, p_{1l}, p_{2l-1}, p_{2l}, \dots, p_{l-2l-1}, p_{l-2l}, p_{l-1}; p_1+2p_{2l-2}+p_{l-1l}, \\ p_{12}+2p_{3l-2}+p_{l-1l}, \dots, p_{1l-3}+2p_{l-2}+p_{l-1l}, p_2+2p_{3l-2}+p_{l-1l}, \\ p_{23}+2p_{4l-2}+p_{l-1l}, \dots, p_{2l-3}+2p_{l-2}+p_{l-1l}, \dots, p_{l-3}+2p_{l-2}+p_{l-1l}\},\end{gathered}$$

$$\begin{gathered}\Phi^+ \setminus \Phi_l^+ = \{p_{1l}, p_{2l}, \dots, p_{l-2l}; p_1+2p_{2l-2}+p_{l-1l}, p_{12}+2p_{3l-2}+p_{l-1l}, \dots, \\ p_{1l-3}+2p_{l-2}+p_{l-1l}, p_2+2p_{3l-2}+p_{l-1l}, p_{23}+2p_{4l-2}+p_{l-1l}, \dots, \\ p_{2l-3}+2p_{l-2}+p_{l-1l}, \dots, p_{l-3}+2p_{l-2}+p_{l-1l}; p_{1l-2}+p_l, p_{2l-2}+p_l, \\ \dots, p_{l-3l-2}+p_l, p_{l-2}+p_l, p_l\}.\end{gathered}$$

For all $\beta \in \Phi^+ \setminus \Phi_{l-1}^+$ or $\beta \in \Phi^+ \setminus \Phi_l^+$, we have shape$(\beta) = 1p_{l-1}$ or shape$(\beta) = 1p_l$ respectively, and hence U is an abelian group and there is only one chief factor. The dimensions of the modules are now easily computable. The proposition is proved.

We now compile Table 7. In the first column of the table, indicate the chief factors $V_S = V_{jp_k}$ occurring in the unipotent radical of each parabolic maximal subgroup P_k in a classical group. Recall that the module V_S is isomorphic to the direct sum of root subgroups X_β, which are one-dimensional weight spaces of weight β. In the second column, we write down all these weights β.

We give all remaining information about the chief factors of parabolic maximal subgroups in the classical groups in the last combined Table 8. In the first column we indicate $G \in \{A_l(q), B_l(q), C_l(q), D_l(q)\}$; in the second column, write down the number of the vertex k that is removed from the Dynkin diagram for obtaining a parabolic maximal subgroup $P = P_k$ in G; in the third column of the table, for P_k, we point out all chief factors $V_S = V_{jp_k}$ occurring in its unipotent radical. In the fourth column, we write down the highest weight of the irreducible KL-module V_S; and, finally, the last column will contain the dimension of V_S.

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