

Towards matrix model representation of HOMFLY polynomials

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Abstract

We investigate possibilities of generalizing the TBEM eigenvalue matrix model, which represents the non-normalized colored HOMFLY polynomials for torus knots as averages of the corresponding characters. We look for a model of the same type, which is a usual Chern-Simons mixture of the Gaussian potential, typical for Hermitean models, and the sine Vandermonde factors, typical for the unitary ones. We mostly concentrate on the family of twist knots, which contains a single torus knot, the trefoil. It turns out that for the trefoil the TBEM measure is provided by an action of Laplace exponential on the Jones polynomial. This procedure can be applied to arbitrary knots and provides a TBEM-like integral representation for the $N = 2$ case. However, beyond the torus family, both the measure and its lifting to larger N contain non-trivial corrections in $\hbar = \log q$. A possibility could be to absorb these corrections into a deformation of the Laplace evolution by higher Casimir and/or cut-and-join operators, in the spirit of Hurwitz τ -function approach to knot theory, but this remains a subject for future investigation.

1 Introduction

Knot polynomials [1] are examples of the Hurwitz τ -function [2], a new and intriguing generalization of the free-fermion [3] KP/Toda τ -functions, probably related to non-Abelian τ -functions of [4]. As such they should possess a number of different realizations: as functional integrals in free field and topological theories [5], as matrix models of the ordinary and Kontsevich types [6], as various W -representations [7] a la [8, 9, 10]. While the first of these representations is well known: knot polynomials are Wilson line averages in Chern-Simons theory [11, 12] and/or results of R -matrix (modular group) evolution of conformal blocks [12, 13, 14], all the other realizations are more-or-less available only for the very specific class of torus knots and links: this story is mostly around the Rosso-Jones formula [15]. In particular, the matrix model representation is known only for the unknot (Chern-Simons partition function) [16]

$$Z_{CS} = \int \prod_{i < j}^N \sinh^2(u_i - u_j) \prod_{i=1}^N \exp\left(-\frac{u_i^2}{2\hbar}\right) du_i \quad (1)$$

and for arbitrary $[m, n]$ torus link/knot [17, 18]:

$$H_R^{[m, n]}(q|A) \Big|_{q=e^{\hbar}, A=q^N} \sim \int \chi_R(e^U) \prod_{i < j}^N \sinh\left(\frac{u_i - u_j}{m}\right) \sinh\left(\frac{u_i - u_j}{n}\right) \prod_{i=1}^N \exp\left(-\frac{u_i^2}{mn\hbar}\right) du_i \quad (2)$$

(here and everywhere in this paper knot polynomials are non-normalized).

However, despite being now available only for torus knots, all such realizations should exist for an arbitrary family of knots, what is strongly supported by the overwhelming success of the evolution method [19, 20]. Still, it is a long-standing problem to generalize (2), to begin with, beyond the very special family of torus knots. This is the goal of the present letter to make a step towards this generalization. Though the final answer remains not yet reached, we realize a few essential properties of a possible final answer for the knot matrix model.

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2 Summary

We are looking for an answer for the HOMFLY polynomial in the matrix model form

$$H_R^\mathcal{L} \sim \frac{\int \chi_R(e^U) \prod_{i<j}^N \mu_\gamma^\mathcal{L}(u_i - u_j|\tau) \sinh(u_i - u_j) \prod_{i=1}^N \exp\left(-\frac{u_i^2}{2\gamma\hbar}\right) du_i}{\int \prod_{i<j}^N \mu_\gamma^\mathcal{L} \sinh(u_i - u_j|\tau) \prod_{i=1}^N \exp\left(-\frac{u_i^2}{2\gamma\hbar}\right) du_i} \quad (3)$$

where \sim means a factor that depends on q , N and representation R in a controllable way. γ is a yet unknown constant, and we choose an ansatz for the measure to depend on N only through a function $\tau(N, \hbar)$. We propose to construct such generalization of the TBEM model to the twist knots in a few steps.

- First, one considers the case of $N = 2$: then the question is, what is the relevant integral representation of the Jones polynomial. The answer is universal: since the inverse of the integral transform

$$F(\rho) = e^{-\frac{\rho^2}{2\hbar}} \int_{-\infty}^{\infty} \mu(u) \cdot \sinh\left(\frac{\rho u}{\hbar}\right) \exp\left(-\frac{u^2}{2\gamma\hbar}\right) du \quad (4)$$

(we need it in application to odd functions) is

$$\mu(u) = e^{-\frac{\gamma\hbar}{2}\partial_u^2} F\left(\frac{u}{\gamma}\right) \quad (5)$$

the measure in the matrix-model integral is made from the Laplace evolution of the Jones polynomial $\mathcal{J}^\mathcal{L}(\rho|\hbar)$ for the link/knot \mathcal{L} , rewritten in appropriate variables (ρ, \hbar) . After making a substitution $\rho = \frac{u}{\gamma}$, it is

$$\boxed{\mu_\gamma^\mathcal{L}(u|\hbar) = e^{-\frac{\gamma\hbar}{2}\partial_u^2} \mathcal{J}^\mathcal{L}\left(\frac{u}{\gamma} \mid \hbar\right)} \quad (6)$$

This, however, gives the answer without dividing by the normalization integral as in (3), which leads to the normalization factor in (3).

- Usually the Jones polynomial $J_r(q)$ is a function of q and of the spin $(r-1)/2$ of representation of $SU(2)$. Eq.(6) deals with $\mathcal{J}(\rho|\hbar) = J_{\rho/\hbar}(e^\hbar)$ obtained from $J_r(q)$ by the substitution $r \rightarrow \rho = r\hbar$, well familiar from the study of Kashaev limit [21] and Hikami invariants [22], and we denote the function of these new variables (ρ, \hbar) by the calligraphic letter, $\mathcal{J}(\rho|\hbar)$ which implies some analytic continuation in the discrete index r described below. In fact, since we will be not able to perform an exact integration in (3), we study series in \hbar .
- The next step is equally universal: the N -fold integral for HOMFLY polynomial is made from this measure by direct analogue of (2):

$$H_R^\mathcal{L} = \frac{\langle \chi_R \rangle^\mathcal{L}}{\langle 1 \rangle^\mathcal{L}} \quad (7)$$

where

$$\boxed{\langle G \rangle^\mathcal{L} = \int G(e^U) \prod_{i<j}^N \left(\mu_\gamma^\mathcal{L}(u_i - u_j|\tau) \cdot \sinh(u_i - u_j) \right) \prod_{i=1}^N \exp\left(-\frac{u_i^2}{2\gamma\hbar}\right) du_i} \quad (8)$$

and $\mu_\gamma^\mathcal{L}(u|\tau)$ is an *odd* function of the integration variable u .

- However, it is of course impossible to reconstruct HOMFLY polynomial from Jones in a universal way: something in this reconstruction should depend on the *type* of the knot. For the two simplest families, of torus and twist knots the difference is basically in the choice of the evolution parameter τ :

$$\begin{aligned} \text{for torus links/knots} \quad \tau &= \hbar = \log q \\ \text{for twist knots} \quad \tau &= \frac{1}{2}N\hbar = \log A^{1/2} \end{aligned} \quad (9)$$

what is in perfect accordance with what we know from the study evolution method in [20].

It is an intriguing question, what happens for other families. But now the way is open to study this kind of problems – which look very promising.

3 The role of γ

What happens in the case of *torus* links/knots, is that there is an additional great simplification: one can choose auxiliary parameter γ in such a way, that the result of Laplace evolution in (6) gets \hbar -independent and actually the measure gets nearly trivial – namely, reduces to that in (2). The choice is clear from (2)

$$\gamma^{[m,n]} = -mn \quad (10)$$

(note the sign minus indicating a non-naive choice of integration contour, or analytical continuation of the answer, if one prefers, which implicit in (2)).

It is an open question for us, what is the meaning of this spectacular possibility, and if some counterpart of it exists in general. Even for the twist knots we have not yet resolved this problem.

Now we provide some evidence in support of above claims. We discuss the family of twist knots, following the description in [20, s.5.2], which we assume the reader to be familiar with. In this brief presentation all the torus links/knots will be represented by a single trefoil, which is also a member of the twist family. All the claims, illustrated by this is example, are actually true for entire torus family.

4 Jones polynomials

According to general principles of the link differential calculus [23, 24, 25, 26], the HOMFLY polynomial is decomposed into a sum of products of the quantities $\{Aq^a\} = Aq^a - A^{-1}q^{-a}$. In particular, for the Jones polynomial there is usually a hypergeometric type expansion [27], which is especially simple for the twist knots [28, 20]:

$$J_{[r-1]}^{(k)}(q) = \sum_{s=0}^{r-1} F_s^{(k)} \prod_{j=-s}^s \{q^{r+j}\} = [r] + [r-1][r][r+1]F_1^{(k)} + [r-2][r-1][r][r+1][r+2]F_2^{(k)} + \dots \quad (11)$$

where the square brackets denote quantum numbers $[r] = \{q^r\}/\{q\}$. Note that we shifted the labeling of representations by one to simplify the formulas below. The coefficient functions $F_s^{(k)}$, polynomials in $A = q^N = e^{N\hbar}$ and $q = e^{\hbar}$ for the twist knot number k are found in sec.5.2 of [20]:

$$F_s^{(k)} = q^{s(s-1)/2} A^s \sum_{j=0}^s \frac{(-)^j [s]!}{[j]![s-j]!} \frac{(Aq^{j-1})^{2jk} \{Aq^{2j-1}\}}{\prod_{i=j-1}^{s+j-1} \{Aq^i\}} = (-k)^s + O(\hbar) \quad (12)$$

The figure eight knot 4_1 corresponds to $k = -1$, the trefoil 3_1 to $k = 1$, unknot arises at $k = 0$. The Rolfsen table notation [29] is $(2|k| + 2)_1$ for negative k and $(2k + 1)_2$ for positive k (for 3_1 the Rolfsen labeling is not smooth: 3_2 is actually 3_1 , since it is the only knot with plane projection having only three intersections).

Since

$$\prod_{j=-s}^s \{q^{r+j}\} = \sum_{j=0}^s (-)^j \frac{[2s+1]!}{[j]![2s+1-j]!} \{q^{(2s+1-2j)r}\} \quad (13)$$

one gets for the measure (after the substitution $s = p + j$)

$$\mu_\gamma^{\mathcal{L}}(u|\hbar) = \frac{1}{\sinh u} \sum_{p=0}^{\infty} q^{-(2p+1)^2/2\gamma} \sinh\left(\frac{(2p+1)u}{\gamma}\right) \sum_{j=0}^{\infty} (-)^j \frac{[2p+2j+1]!}{[j]![2p+j+1]!} F_{p+j}^{(k)} \Big|_{A=q^2} \quad (14)$$

This q -hypergeometric function is equivalent to (6) and along with (9) gives the answer for the matrix integral (3).

5 Comments

5.1 Jones polynomials in variables (ρ, \hbar)

Formulas similar to (14) are not that simple to deal with. When performing checks for the matrix model, we used the \hbar -expansion instead. Let us see how these checks are done.

One of the possibilities is to use (14) directly. Say, in the leading order

$$\sum_{j=0}^{\infty} (-)^j \frac{(2p+2j+1)!}{j!(2p+j+1)!} (-k)^{p+j} = (-k)^p \cdot {}_2F_1(p+1, p+3/2; 2p+2; -4k) = \frac{2 \cdot (-4k)^p}{\sqrt{1-4k} (1 + \sqrt{1-4k})^{2p+1}} \quad (15)$$

and the sum in (14) is easily calculated:

$$\mu_{(0)}(u) = \frac{\sinh\left(\frac{u}{\gamma}\right)}{1 + 4k \sinh^2\left(\frac{u}{\gamma}\right)} \quad (16)$$

Another, simpler possibility is to use formula (6). Indeed, let us make a substitution $r \rightarrow \rho = r\hbar$:

$$[r] = \frac{2 \sinh \rho}{\{q\}}, \quad \{q\}[r-1] = 2 \sinh(\rho - \hbar) \quad \text{etc} \quad (17)$$

Then, from (11)

$$\begin{aligned} \frac{1}{2} \{q\} \mathcal{J}^{(k)}(\rho, \hbar) &= \sinh \rho \left(1 + 4F_1^{(k)} \sinh(\rho - \hbar) \sinh(\rho + \hbar) + \right. \\ &\quad \left. + 16F_2^{(k)} \sinh(\rho - 2\hbar) \sinh(\rho - \hbar) \sinh(\rho + \hbar) \sinh(\rho + 2\hbar) + \dots \right) = \\ &= \sinh \rho \sum_{s=0}^{\infty} (-4k \sinh^2 \rho)^s \left\{ 1 + \hbar(k+1) \left(\frac{2s(s+2)}{3} - \frac{s(s-1)}{6k} \right) + O(\hbar^2) \right\} = \\ &= \frac{\sinh \rho}{1 + 4k \sinh^2 \rho} - 8k(k+1)\hbar \left(1 + \frac{2(2k+1)}{3} \sinh^2 \rho \right) \left(\frac{\sinh \rho}{1 + 4k \sinh^2 \rho} \right)^3 + O(\hbar^2) \end{aligned} \quad (18)$$

Similarly, the \hbar^2 -correction is given by

$$\begin{aligned} \frac{1}{30} \sum_{s=0}^{\infty} (-k)^{s-2} \left(2 \sinh(\rho) \right)^{2s+1} &\left(\frac{1}{24} s(s-1)(s-2)(5s+1) - \frac{5}{4} s^2(s+3)(s-1)k + \frac{5}{24} s(s^3 + 34s^2 + 71s - 10)k^2 + \right. \\ &\quad \left. + 5s^2(s+2)(s+3)k^3 + \frac{2}{3} s(s+3)(5s^2 + 9s + 1)k^4 \right) - \frac{1}{3} \sum_{s=0}^{\infty} s(s+1)(2s+1)(-k)^s \left(2 \sinh(\rho) \right)^{2s-1} = \\ &= \frac{k \sinh \rho}{15 \left(1 + 4k \sinh^2 \rho \right)^5} \left((4k^4 + 10k^3 + 10k^2 + 5k + 1)kx^8 + 2(k+1)(12k^3 + 13k^2 + 7k - 2)x^6 + \right. \\ &\quad \left. + 5(2k-1)(6k^2 + 7k + 5)x^4 - 20(k+1)(2k+1)x^2 + 60 \right) \end{aligned} \quad (19)$$

where $x = 2 \sinh \rho$. We do not write down the next corrections, since they are too long.

5.2 Measure from the Laplace evolution

Applying now (6), one obtains:

$$\begin{aligned} \mu_{\gamma}^{(k)} &= \frac{\sinh\left(\frac{u}{\gamma}\right)}{1 + 4k \sinh^2\left(\frac{u}{\gamma}\right)} - \\ &- \frac{\gamma \hbar}{2} \frac{\partial^2}{\partial u^2} \frac{\sinh\left(\frac{u}{\gamma}\right)}{1 + 4k \sinh^2\left(\frac{u}{\gamma}\right)} - 8k(k+1)\hbar \left(1 + \frac{2(2k+1)}{3} \sinh^2\left(\frac{u}{\gamma}\right) \right) \left(\frac{\sinh\left(\frac{u}{\gamma}\right)}{1 + 4k \sinh^2\left(\frac{u}{\gamma}\right)} \right)^3 + O(\hbar^2) = \\ &= \left(1 + \frac{(1-24k)\hbar}{2\gamma} + O(\hbar^2) \right) \left\{ \frac{\sinh\left(\frac{u}{\gamma}\right)}{1 + 4k \sinh^2\left(\frac{u}{\gamma}\right)} - \hbar \left(\frac{\sinh\left(\frac{u}{\gamma}\right)}{1 + 4k \sinh^2\left(\frac{u}{\gamma}\right)} \right)^3 \times \right. \\ &\quad \left. \times \left(\left(\frac{16k(k+1)(2k+1)}{3} + \frac{192k^3}{\gamma} \right) \sinh\left(\frac{u}{\gamma}\right)^2 + \left(8k(k+1) + \frac{16k(7k-1)}{\gamma} \right) \right) + O(\hbar^2) \right\} \end{aligned} \quad (20)$$

We definitely calculated a lot more corrections which we used to make our checks.

5.3 The case of torus knots ($k = 1$)

Both coefficients in the last line of (20) vanish in the case of trefoil: for $k = 1$ and $\gamma = -6$. In fact, this remains true for all higher \hbar -corrections: **in the case of trefoil and of other torus knots/links the evolution operator $e^{-\frac{\gamma\hbar}{2}\partial_\rho^2}$ (6) converts their Jones polynomials (expressed via ρ -variable) into \hbar -independent quantities** (modulo overall u -independent normalization factor).

As already mentioned we use the trefoil to illustrate the *generic* feature of the torus family. The peculiarities are two: for $k = 1$ the leading-order measure can be rewritten in the form of (2):

$$\hbar\mathcal{J}^{(1)} = \frac{\sinh \rho}{1 + 4 \sinh^2 \rho} + O(\hbar) \quad (21)$$

and

$$\boxed{\frac{\sinh \rho}{1 + 4 \sinh^2 \rho} = \frac{\sinh(2\rho) \sinh(3\rho)}{\sinh(6\rho)}} \quad (22)$$

Moreover, this answer is exact: all \hbar -corrections to (21) are *exactly* eliminated by the action of Laplace exponential. This latter property is true only for $k = 1$, and also depends on the clever choice of $\gamma = -mn = -6$. More accurately, for such γ all the corrections can be absorbed into overall u -independent normalization coefficient in front of $\mu(u)$, which drops away from the ratio of integrals and do not affect the averages.

Extension of this result to other torus knots is not at all trivial. Already for the next 2-strand knot $[2, 5] = 5_1$ the relevant analogue of identity (22) is

$$\frac{\sinh(4\rho) \sinh(6\rho)}{\sinh(12\rho)} = \frac{\sinh \rho}{1 + 12 \sinh^2 \rho + 16 \sinh^4 \rho} = \frac{x}{2} \left(1 - 3 \cdot x^2 + 8 \cdot x^4 - 21 \cdot x^6 + \dots \right) \quad (23)$$

and to obtain the r.h.s. from Jones polynomial one needs to know the large- r asymptotics of the coefficients $g_{r,j}$ in eq.(64) of [25]:

$$g_{r,1}^{[2,5]} \sim 3 \cdot r, \quad g_{r,2}^{[2,5]} \sim 8 \cdot \frac{r^2}{2!}, \quad g_{r,1}^{[2,5]} \sim 21 \cdot \frac{r^3}{3!}, \quad \dots \quad (24)$$

Again after that one can adjust $\gamma = -10$ so that all \hbar -corrections are eliminated by the action of exponentiated Laplacian.

5.4 Checks for $N = 2$

Despite we derived the (formal series for) the measure, starting from Jones polynomial in the variables (ρ, \hbar) , we obtained the answer, which can be used in (7) to evaluate HOMFLY polynomials in concrete representations $[r - 1]$, starting from the fundamental one.

At $N = 2$ we can calculate Jones polynomials, but in variables (r, \hbar) , i.e. check that (7) reproduces (11). More precisely, this works up to a factor: what is reproduced is the series (11) times $q^{-\alpha(r^2-1)}$. Comparing this factor with that in front of the integral transform (4), it is easy to anticipate $\alpha = -\frac{1}{2}\gamma$.

The measure $\langle \dots \rangle^{(k)}$ reproducing (11) should satisfy

$$q^{-\alpha(r^2-1)} J_r^{(k)}(q) = \left\langle \frac{\sinh(ru)}{\sinh(u)} \right\rangle^{(k)} = \sum_{p=0}^{\infty} c_{r,p} \langle u^{2p} \rangle^{(k)} \quad (25)$$

– and the claim is that it is indeed given by (7) and (20). The coefficients in (25) are

$$\begin{aligned}
c_{r,0} &= r, \\
c_{r,1} &= \frac{1}{6}r(r^2 - 1), \\
c_{r,2} &= \frac{1}{360}r(r^2 - 1)(3r^2 - 7), \\
c_{r,3} &= \frac{1}{15120}r(r^2 - 1)(3r^4 - 18r^2 + 31), \\
c_{r,4} &= \frac{1}{1814400}r(r^2 - 1)(5r^6 - 55r^4 + 239r^2 - 381), \\
&\dots
\end{aligned} \tag{26}$$

The measure and the averages depend on $q = e^{\hbar}$, and we expand them in series in \hbar :

$$\langle u^{2p} \rangle^{(k)} = \sum_{j=0}^{\infty} \gamma_{p,j}^{(k)} \hbar^{p+j} \tag{27}$$

Then we compare the expansion $J_r^{(k)} = \sum_{i=0}^{\infty} J_{r,i}^{(k)} \hbar^i$, obtained directly from (11), with $\sum_{p,j}^{\infty} c_{r,p} \gamma_{p,j}^{(k)} \hbar^{p+j}$. Despite the latter one is the double sum, one can use r -dependence to extract from this comparison all the coefficients $\gamma_{p,j}^{(k)}$. This gets clear from looking at the first terms of the double expansion:

$$\begin{aligned}
0 &= \sum_{p,j}^{\infty} c_{r,p} \gamma_{p,j}^{(k)} \hbar^{p+j} - J_r^{(k)} = \\
&= r \left(\gamma_{0,0}^{(k)} - 1 \right) + \frac{1}{6} \hbar r^3 \left(\gamma_{1,0}^{(k)} - 6\alpha \right) + \hbar r \left(\gamma_{0,1}^{(k)} - \frac{1}{6} \gamma_{1,0}^{(k)} + \alpha \right) + \frac{1}{120} \hbar^2 r^5 \left(\gamma_{2,0}^{(k)} - 60\alpha^2 \right) + \\
&+ \frac{1}{6} \hbar^2 r^3 \left(\gamma_{1,1}^{(k)} - \frac{1}{6} \gamma_{2,0}^{(k)} - 1 + 24k + 6\alpha^2 \right) + \hbar^2 r \left(\gamma_{0,2}^{(k)} - \frac{1}{6} \gamma_{1,1}^{(k)} + \frac{7}{360} \gamma_{2,0}^{(k)} + \frac{1}{6} - 4k - \frac{1}{2} \alpha^2 \right) + \dots
\end{aligned} \tag{28}$$

The first item at each bracket is expressed through the others, already determined at the previous stage. In this way we obtain:

$$\gamma_{p,0}^{(k)} = \frac{(4\alpha)^p \Gamma(p + \frac{3}{2})}{\Gamma(\frac{3}{2})},$$

$$\gamma_{p,1}^{(k)} = \frac{(1 - 24k + 4\alpha^2)p}{24\alpha^2} \cdot \frac{(4\alpha)^{p+1} \Gamma(p + \frac{3}{2})}{\Gamma(\frac{3}{2})},$$

$$\gamma_{p,2}^{(k)} = \frac{3(1 - 240k + 1920k^2 + 16\alpha^4)p(p-1) + 40(1 - 24k)\alpha^2 p(p+1) + 2880k(k+1)\alpha p}{96 \cdot 60 \alpha^4} \cdot \frac{(4\alpha)^{p+2} \Gamma(p + \frac{3}{2})}{\Gamma(\frac{3}{2})},$$

$$\begin{aligned}
\gamma_{p,3}^{(k)} &= \left((1 - 2184k + 67200k^2 - 322560k^3 + 64\alpha^6)p(p-1)(p-2) + 28 \left((1 - 240k + 1920k^2)\alpha^2 + 4(1 - 24k)\alpha^4 \right) (p+2)p(p-1) + \right. \\
&+ 6720k(k+1)(64k - 7)\alpha p(p-1) - 26880k(k+1)\alpha^3 p(p+1) - 53760k(k^2 + 4k + 1)\alpha^2 p \left. \right) \frac{1}{384 \cdot 840 \alpha^6} \cdot \frac{(4\alpha)^{p+3} \Gamma(p + \frac{3}{2})}{\Gamma(\frac{3}{2})}, \\
&\dots
\end{aligned} \tag{29}$$

With our formulas one can check that these parameters are indeed reproduced by (7), to the accuracy of the first three orders of \hbar -expansion.

5.5 The cases of $N = 3$ and $N = 4$

The same check can be performed for higher $N > 2$, making use of the measure (8). This time HOMFLY polynomials could be reproduced only if τ in (8) is not just \hbar , but rather $\frac{N\hbar}{2}$. We now provide a little more details.

Let us fix an ansatz for $\tau = T\hbar$, where T is a constant that we are going to determine, i.e. T is the coefficient in front of the \hbar -correction to the measure:

$$\prod_{1 \leq i < j \leq N} \sinh^2(u_{ij}) \left(\mu_{(0)}(u_{ij}) + T\hbar \mu_{(1)}(u_{ij}) + T^2 \hbar^2 \mu_{(2)}(u_{ij}) + O(\hbar^3) \right) \quad (30)$$

with

$$\mu_{(0)} = \frac{\sinh\left(\frac{u}{\gamma}\right)}{\sinh(u) \left(1 + 4k \sinh^2\left(\frac{u}{\gamma}\right)\right)} \quad (31)$$

etc. The same measure (30) is used in the numerator and denominator.

The ratios of matrix model integrals and the corresponding HOMFLY polynomials \mathfrak{R} in the order \hbar^3 for the fundamental representation ($r = 2$) is (up to a power of q)

$$\mathfrak{R} \sim 1 + 24\hbar^3(T-1)f(k, \gamma) \quad (32)$$

for $SU(2)$, and

$$\mathfrak{R} \sim 1 + 32\hbar^3(2T-3)f(k, \gamma) \quad (33)$$

for $SU(3)$. For $N = 4$ the factor is $T - 2$, so in general it is probably $T - \frac{1}{2}N$.

The values of \mathfrak{R} at $A = q^3$ and $A = q^4$ in representations $[r-1] = [1]$ and $[2]$ are equal to:

$$\begin{array}{ll} N = 2, [1]: & q^{2\gamma} \quad 1 - \frac{24k\hbar^3}{\gamma} (14k - 2 + \gamma + \gamma k)(T-1) + O(\hbar^4) \\ N = 2, [2]: & q^{6\gamma} \quad 1 - \frac{64k\hbar^3}{\gamma} (14k - 2 + \gamma + \gamma k)(T-1) + O(\hbar^4) \\ \hline N = 3, [1]: & q^{3\gamma} \quad 1 - \frac{64k\hbar^3}{\gamma} (14k - 2 + \gamma + \gamma k) \left(T - \frac{3}{2}\right) + O(\hbar^4) \\ N = 3, [2]: & q^{8\gamma} \quad 1 - \frac{160k\hbar^3}{\gamma} (14k - 2 + \gamma + \gamma k) \left(T - \frac{3}{2}\right) + O(\hbar^4) \\ \hline N = 4, [1]: & q^{4\gamma} \quad 1 - \frac{120k\hbar^3}{\gamma} (14k - 2 + \gamma + \gamma k)(T-2) + O(\hbar^4) \\ N = 4, [2]: & q^{10\gamma} \quad 1 - \frac{288k\hbar^3}{\gamma} (14k - 2 + \gamma + \gamma k)(T-2) + O(\hbar^4) \\ & \dots \end{array} \quad (34)$$

Here in the second column we put the normalization factor that differs the matrix model integral and the HOMFLY polynomial in the topological framing.

Thus, the answer looks like

$$\boxed{\mathfrak{R} = q^{r(N+r-1)\gamma} \left[1 - \frac{8r(N-1)(N+r)k\hbar^3}{\gamma} (14k - 2 + \gamma + \gamma k) \left(T - \frac{N}{2}\right) + O(\hbar^4) \right]} \quad (35)$$

Since the coefficient in the brackets vanishes for

$$\gamma \stackrel{?}{=} -\frac{2(7k-1)}{k+1} \quad (36)$$

one may think that there is no need to require $T = \frac{N}{2}$ (though for the figure eight knot this does not work anyway). However, this does not work already in the next order \hbar^4 : this already does not work, the correction

does not vanish for this choice of γ . Note that only for $k = 1$ this choice ($\gamma = -6$) makes the correction to the measure vanishing as well.

In the cases of higher $N > 2$ one can also consider more complicated representations than just symmetric ones. For instance, for $N = 3$ one can compare the result of matrix model calculation with the HOMFLY polynomial at $A = q^3$ in representation [21] which is known for the trefoil and for the figure eight. We assume that, at least, in the leading orders the HOMFLY polynomial for other twist knots is given by the same functions $F_p^{(k)}$. Then, the result in this case reads

$$\mathfrak{R} = q^{9\gamma} \left(1 - \frac{72k(14k - 2 + \gamma + \gamma k)(2T - 3)}{\gamma} \hbar^3 - \right. \\ \left. -24k(2T - 3) \left[(2k^2 + 15k + 1)(2T + 3) + \frac{8T(k + 1)(31k - 4)}{\gamma} + \frac{(828k^2 - 186k + 6)(2T - 3)}{\gamma^2} \right] \hbar^4 + O(\hbar^5) \right) \quad (37)$$

and one can conclude that the formulas are still correct in this case, supporting the idea to use the same $F_p^{(k)}$ for all representations (see sec.4.3 of ref.[25] for more careful formulation of this hypothesis). This example is just the simplest illustration of power of the matrix model approach: even when being not completed, it already provides new results, which are very difficult to get by other methods.

5.6 The \hbar^5 -corrections: violation of universality

Unfortunately, in higher orders of \hbar the described procedure does not give a complete answer: starting from the order \hbar^5 one needs to correct the matrix model result in order to reproduce the right value of the knot polynomial. These corrections can be absorbed into the normalization factor in (3): they have a rather simple dependence on the representation and N . For instance, in the order \hbar^5 they are:

$$\mathfrak{R} = q^{r(N+r-1)\gamma} \left[1 + 2(3N - 4)(N - 2) \left(2N\kappa_R + r(N^2 - r) \right) U(k, \gamma) \frac{\hbar^5}{\gamma^3} + O(\hbar^6) \right] \quad (38)$$

where the quantity

$$U(k, \gamma) = k \left(4\gamma^3 k^2 + 3\gamma^3 k - \gamma^3 + 1872k^3 - 624k^2 + 48 \right) \quad (39)$$

does not depend on R and N , and at $k = 1$ factorizes as

$$U(1, \gamma) = 6(\gamma + 6)(\gamma^2 - 6\gamma + 36) \frac{\hbar^5}{\gamma^3} \quad (40)$$

Thus, as before, the correct behaviour of the trefoil is guaranteed at $\gamma = -6$ in higher orders as well.

The coefficient κ_R in (38) is the eigenvalue of the second Casimir operator:

$$\kappa_R = \sum_{i, j \in R} (j - i) \quad (41)$$

where R denotes the Young diagram corresponding to the representation R and the sum goes over the boxes of this Young diagram with coordinates (i, j) . This quantity is also the eigenvalue of the simplest cut-and-join operator $\widehat{W}_{[2]}$ [30]. One can expect that the higher orders in \hbar could be described by higher Casimir or cut-and-join operators [31] in spirit of [7].

6 Conclusion

In this letter we developed a systematic approach to study the TBEM-like integral (matrix model) representations of knot polynomials.

The starting point is the Jones polynomial as a function of representation variable: then the action of exponentiated Laplace operator immediately provides a measure for an integral representation of the original Jones polynomial – which in the case of the trefoil is exactly the right TBEM measure for $N = 2$. This, however, is true only if the evolution "time" is appropriately adjusted ($\gamma = -6$). Two immediate questions here are: how this works for other torus knots, and what happens, if one deforms the Laplace evolution.

The next step is lifting the measure from $N = 2$ to higher N . The natural prescription (3) is in fact equivalent to promoting the 2-particle Calogero-Ruijsenaars evolution to the N -particle one. Again, a natural question is what happens, if one allows higher Hamiltonians to contribute.

We demonstrated that all these questions can indeed be relevant, because the above two-step procedure works perfectly only for the trefoil: the Laplace evolution and its ordinary lifting to higher N is indeed equivalent to the TBEM model (though an exact proof is still needed even in this case).

Already for the family of twist knots there are corrections which clearly exhibit a clever representation dependence to be described by some kind of a deformation of the Laplace evolution (by higher Hamiltonians, i.e. Casimirs, or, perhaps, by more general cut-and-join operators [31]). Unfortunately, the technique of \hbar -expansion which we used in this letter (and in a closely related investigation of the Hurwitz τ -function structure of colored HOMFLY polynomials and superpolynomials in [7]) is not sufficient to answer these questions.

More powerful methods of group, matrix model, conformal and integrability theories should now be applied to attack this very promising problem. Among immediate topics to study are the clearly seen relations to the volume conjecture and Hikami invariants [21, 22, 32].

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