ON PALINDROMIC WIDTH OF CERTAIN EXTENSIONS AND QUOTIENTS OF FREE NILPOTENT GROUPS

VALERIY G. BARDAKOV AND KRISHNENDU GONGOPADHYAY

Abstract. In [3] the authors provided a bound for the palindromic width of free abelian-by-nilpotent group AN_n of rank n and free nilpotent group $\mathrm{N}_{n,r}$ of rank n and step r. In the present paper we study palindromic widths of groups $\widetilde{AN}_n$ and $\widetilde{\mathrm{N}}_{n,r}$. We denote by $\widetilde{G}_n = G_n/\langle\langle x_1^2, \ldots, x_n^2\rangle\rangle$ the quotient of group $G_n = \langle x_1, \ldots, x_n\rangle$, which is free in some variety by the normal subgroup generated by $x_1^2, \ldots, x_n^2$. We prove that the palindromic width of the quotient $\widetilde{AN}_n$ is finite and bounded by $3n$. We also prove that the palindromic width of the quotient $\widetilde{\mathrm{N}}_{n,2}$ is precisely $2(n-1)$. We improve the lower bound of the palindromic width for $\mathrm{N}_{n,r}$. We prove that the palindromic width of $\mathrm{N}_{n,r}$, $r \geq 2$ is at least $2(n-1)$. We also improve the bound for palindromic widths of free metabelian groups. We prove that the palindromic width of free metabelian group of rank n is at most $4n-1$.

1. Introduction

Let S be a set of generators of a group G. A reduced word w in the alphabet $S^{\pm 1}$ is called a *palindrome* if w reads the same left-to-right and right-to-left. An element g of G is called a *palindrome* if g can be represented by some word w that is a palindrome in the alphabet $S^{\pm 1}$. We denote the set of all palindromes in G by $\mathcal{P} = \mathcal{P}(S)$. Evidently, the set $\mathcal{P}$ generates G. Then any element $g \in G$ is a product of palindromes

$$g = p_1 p_2 \ldots p_k.$$

The minimal k with this property is called the *palindromic length* of g and is denoted by $l_{\mathcal{P}}(g)$. The *palindromic width* of G is given by

$$\mathrm{pw}(G,S) = \mathrm{wid}(G,\mathcal{P}) = \sup_{g \in G} l_{\mathcal{P}}(g).$$

When there is no confusion about the underlying set of generators S, the palindromic width with respect to S is simply denoted by $\mathrm{pw}(G)$. In analogy with commutator width of groups, it is an interesting problem to study palindromic width of groups. Palindromes of free groups have been investigated by several people and it has already been useful in studying various aspects of combinatorial group theory and geometry, for example see [4]–[13].

Our primary aim in this article is to investigate the following problem: Let G be a group that is free in some variety $\mathcal{G}$ of groups. Let $G = F_n(\mathcal{G})$ has rank n. Let $X = \{x_1, \ldots, x_n\}$

Date: February 24, 2014.
2000 *Mathematics Subject Classification.* Primary 20F65; Secondary 20D15, 20F18, 20F19.
Key words and phrases. palindromic width, free nilpotent groups.
The authors gratefully acknowledge the support of the Indo-Russian DST-RFBR project grant DST/INT/RFBR/P-137.
Bardakov is partially supported by Laboratory of Quantum Topology of Chelyabinsk State University(Russian Federation government grant 14.Z50.31.0020) .

be a basis of G. Define a group $\widetilde{G}$ that is the quotient group of G by the relations $x_i^2 = 1$, $i = 1, \dots, n$. That is

$$\widetilde{G} = G/\langle\langle x_1^2, \dots, x_n^2 \rangle\rangle,$$

where $\langle\langle x_1^2, \dots, x_n^2 \rangle\rangle$ denotes the normal closure of the set $\{x_1^2, \dots, x_n^2\}$. Let $Y = \{y_1, \dots, y_n\}$ be the image of the generating set $X = \{x_1, \dots, x_n\}$ in $\widetilde{G}$. Then $\widetilde{G} = \langle Y \rangle$.

Problem 1. *What is the value of* $\mathrm{pw}(\widetilde{G}, Y)$*?*

We have obtained progress to this problem for the cases when G is either a free abelian-by-nilpotent group or a free nilpotent group. We prove the following.

Theorem 1.1. *(i) Let AN_n be a free abelian-by-nilpotent group of rank n. Then for $n \geq 2$, $n \leq \mathrm{pw}(\widetilde{AN}_n, Y) \leq 3n$.*

(ii) Further, for a free metabelian group M_n of rank n we have, $n \leq \mathrm{pw}(\widetilde{M}_n, Y) \leq 3n-1$.

The above theorem lead to further examples of finitely generated groups which are not boundedly generated but have finite palindromic width. We recall that a group G is *boundedly generated* if there exist $a_1, \dots, a_k \in G$ such that every element can be expressed as $a_1^{n_1} \dots a_k^{n_k}$ for some integers $n_1, \dots, n_k$, for eg. see [6, 16]. Note that there are free metableian groups those are not boundedly generated. The authors [3] and, Riley and Sale [15] have independently established the finiteness of palindromic width of metabelian groups using different techniques. The authors actually proved a stronger result: any free abelian-by-nilpotent group of rank n has palindromic width at most $5n$, cf. [3, Section 3.4]. In particular, it follows that $\mathrm{pw}(M_n, X) \leq 5n$. In this paper, we further improve the bound. We prove that $\mathrm{pw}(M_n, X) \leq 4n - 1$, see Theorem 3.2.

The palindromic width of a boundedly generated group with respect to the bounded generators $a_1, \dots, a_k$ is at most k. However, for a free group in a variety of boundedly generated groups, generally the bounded generators are not the free set of generators. It is desirable to obtain palindromic widths for these groups with respect to a basis.

Let $\mathrm{N}_{n,r}$ be the free nilpotent group of rank n and of step r. Let X be a basis of $\mathrm{N}_{n,r}$. It is known that finitely generated nilpotent groups are boundedly generated, see [16, Lemma 1.3]. Investigation to obtain precise value of the palindromic width of $\mathrm{N}_{n,r}$ has been initiated by the authors in [3]. The authors have provided a bound for the palindromic width of $\mathrm{N}_{n,r}$: for $n > 1$ and $r \geq 2$, $n \leq \mathrm{pw}(\mathrm{N}_{n,r}, X) \leq 3n$, see [3, Theorem 1.1]. In fact, for $\mathrm{N}_{n,1}$ and $\mathrm{N}_{2,2}$ precise values of the palindromic widths were obtained. For $r = 2$, they further improved the upper bound: $\mathrm{pw}(\mathrm{N}_{n,2}, X) \leq 3(n-1)$. In this paper we investigate the palindromic width of $\mathrm{pw}(\widetilde{\mathrm{N}}_{n,r}, Y)$. We prove the following.

Theorem 1.2. (1) *For $n, r \geq 2$, $n \leq \mathrm{pw}(\widetilde{\mathrm{N}}_{n,r}, Y) \leq 2n$.*

(2) *The palindromic width of $\widetilde{\mathrm{N}}_{n,2}$ (with respect to the set of generators Y) is $2(n-1)$.*

For the group $\widetilde{\mathrm{N}}_{3,2}$, we have proved something more.

Proposition 1.3. *Let z_{ij} denote the commutator $[y_i, y_j]$ in $\widetilde{\mathrm{N}}_{3,2}$. In $\widetilde{\mathrm{N}}_{3,2}$ the only element that can not be expressed as a product of three palindromes is $z_{21}z_{31}z_{32}$. Moreover, $l_{\mathcal{P}}(z_{21}z_{31}z_{32}) = 4$.*

As a corollary to the above theorem, we improve the lower bound of the palindromic width of $\mathrm{N}_{n,r}$ for $r \geq 2$.

Corollary 1. *Let $\mathrm{N}_{n,r}$ be the r-step free nilpotent group of rank $n \geq 2$. Then*

(1) *For $n > 1$ and $r \geq 3$, $2(n-1) \leq \mathrm{pw}(\mathrm{N}_{n,r}, X) \leq 3n$.*
(2) $2(n-1) \leq \mathrm{pw}(\mathrm{N}_{n,2}, X) \leq 3(n-1)$.

After reviewing some preliminary results in Section 2, we investigate palindromic width of $\widetilde{G}$ for G free abelian-by-nilpotent, in Section 3. In particular, we prove Theorem 3.2 in this section. We prove Theorem 1.2 in Section 4. The Section 5 is devoted to the proof of Proposition 1.3.

Acknowledgements. We thank Elisabeth Fink, Andrew Sale and Tim Riley for their comments on this article.

2. Preliminaries

2.1. Palindromes in Groups.

In this Section we collect results that will be useful for us.

Lemma 2.1. [3] *Let $G = \langle X \rangle$ and $H = \langle Y \rangle$ be two groups, $\mathcal{P}(X)$ is the set of palindromes in the alphabet $X^{\pm 1}$, $\mathcal{P}(Y)$ is the set of palindromes in the alphabet $Y^{\pm 1}$. If $\varphi : G \longrightarrow H$ be an epimorphism such that $\varphi(X) = Y$, then*

$$\mathrm{pw}(H) \leq \mathrm{pw}(G).$$

Lemma 2.2. [3, Lemma 2.4] *Let $G = \langle X \rangle$ be a group generated by a set X. Then the following hold.*

(1) *If p is a palindrome, then for m in $\mathbb{Z}$, p^m is also a palindrome.*
(2) *Any element in G which is conjugate to a product of n palindromes, $n \geq 1$, is a product of n palindromes if n is even, and of $n+1$ palindromes if n is odd.*
(3) *Any commutator of the type $[u, p]$, where p is a palindrome is a product of 3 palindromes. Any element $[u, x^\alpha]x^\beta$, $x \in X$, $\alpha, \beta \in \mathbb{Z}$, is a product of 3 palindromes.*
(4) *In G any commutator of the type $[u, pq]$, where p, q are palindromes is a product of 4 palindromes. Any element $[u, px^\alpha]x^\beta$, $x \in X$, $\alpha, \beta \in \mathbb{Z}$, is a product of 4 palindromes.*

Lemma 2.3. [2, Lemma 3] *Let A be a normal subgroup of $G = \langle x_1, x_2, \ldots, x_n \rangle$. If A is abelian or A lies in the second center of G, then every element of $[A, G]$ has the form*

$$[u_1, x_1]\,[u_2, x_2]\,\ldots\,[u_n, x_n] \text{ for some } u_i \in A.$$

2.2. Free Nilpotent Groups.

Let $\mathrm{N}_{n,r}$ be the free r-step nilpotent group of rank n with a basis $x_1, \ldots, x_n$. For example, when $r = 1$, $\mathrm{N}_{n,1}$ is simply the free abelian group generated by $x_1, \ldots, x_n$, so every element of $\mathrm{N}_{n,1}$ can be presented uniquely as

$$g = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

for some integers $\alpha_1, \ldots, \alpha_n$. For $r = 2$, every element $g \in \mathrm{N}_{n,2}$ has the form

$$g = \prod_{i=1}^{n} x_i^{\alpha_i} \cdot \prod_{1 \leq j < i \leq n} [x_i, x_j]^{\beta_{ij}} \tag{2.1}$$

for some integers α_i and β_{ij}, where $[x_i, x_j] = x_i^{-1}x_j^{-1}x_i x_j$ are basic commutators (see [12, Chapter 5]).

For the free nilpotent group $\mathrm{N}_{n,r}$, let $\mathrm{N}'_{n,r}$ be its commutator subgroup. We note the following lemmas that will be used later.

Lemma 2.4. [1, 2] *Any element g in the commutator subgroup $\mathrm{N}'_{n,r}$ can be represented in the form*

$$g = [u_1, x_1]\,[u_2, x_2]\,\dots\,[u_n, x_n],\ \ u_i \in \mathrm{N}_{n,r}.$$

Lemma 2.5. [3, Corollary 1] *The following inequalities hold:*

$$\mathrm{pw}(\mathrm{N}_{n,1}) \leq \mathrm{pw}(\mathrm{N}_{n,2}) \leq \mathrm{pw}(\mathrm{N}_{n,3}) \leq \dots$$

2.2.1. *Normal Forms.* In [3, Lemma 3.3] we have found normal form for palindromes in $\mathrm{N}_{2,2}$. In this subsection we will find normal form for palindromes in $\mathrm{N}_{n,2}$, $n \geq 2$.

Lemma 2.6. *Every palindrome $p \in \mathrm{N}_{n,2}$ has the form*

$$p = x_1^{\alpha_1} \dots x_{i-1}^{\alpha_{i-1}} x_{i+1}^{\alpha_{i+1}} \dots x_n^{\alpha_n} \cdot x_i^{\alpha_0} \cdot x_n^{\alpha_n} \dots x_{i+1}^{\alpha_{i+1}} x_{i-1}^{\alpha_{i-1}} \dots x_1^{\alpha_1},$$

for some integers $\alpha_0, \alpha_1 \dots, \alpha_n$.

Proof. Any palindrome in $\mathrm{N}_{n,2}$ by definition is equal to

$$p = u x_i^{\alpha} \bar{u},\ \ \alpha \in \mathbb{Z},\ \ u \in \mathrm{N}_{n,2},$$

where

$$u = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_k^{\alpha_k},\ \ a_i \in A, \alpha_i \in \mathbb{Z}$$

is a word and

$$\overline{u} = a_k^{\alpha_k} a_{k-1}^{\alpha_{k-1}} \dots a_1^{\alpha_1}$$

is its reverse word.

Let

$$u = \prod_{j=1}^{n} x_j^{\alpha_j} \prod_{1 \leq l < k \leq n} z_{kl}^{\beta_{kl}},\ \text{where } z_{kl} = [x_k, x_l],\ \alpha_j, \beta_{kl} \in \mathbb{Z}$$

be an arbitrary element in $\mathrm{N}_{n,2}$. Then

$$\bar{u} = \prod_{1 \leq l < k \leq n} \bar{z}_{kl}^{\beta_{kl}} \prod_{j=0}^{n-1} x_{n-j}^{\alpha_{n-j}},$$

where $\bar{u}$, $\bar{z}$ are the reverse words of u, z respectively. Since $\bar{z}_{kl} = x_l x_k x_l^{-1} x_k^{-1} = [x_l^{-1}, x_k^{-1}] = [x_l, x_k] = [x_k, x_l]^{-1} = z_{kl}^{-1}$, we have

$$\bar{u} = \prod_{1 \leq l < k \leq n} z_{kl}^{-\beta_{kl}} \prod_{j=0}^{n-1} x_{n-j}^{\alpha_{n-j}}.$$

Using the rules

$$x_i^{\alpha_i} x_j^{\alpha_j} = x_j^{\alpha_j} x_i^{\alpha_i} [x_j, x_i]^{-\alpha_i \alpha_j},\ \ 1 \leq i < j \leq n,$$
$$x_j^{\alpha_j} x_i^{\alpha_i} = x_i^{\alpha_i} x_j^{\alpha_j} [x_j, x_i]^{\alpha_i \alpha_j},\ \ 1 \leq i < j \leq n,$$

we can remove elements $x_i^{\alpha_i}$ in the product $u x_i^{\alpha} \bar{u}$ to the center and cancel the commutators, see [3, Section 3.2] and we see that

$$p = \prod_{\substack{j=1 \\ j \neq i}}^{n} x_j^{\alpha_j} \cdot x_i^{\alpha_0} \cdot \prod_{\substack{j=0 \\ j \neq n-i}}^{n-1} x_{n-j}^{\alpha_{n-j}},\ \ \alpha_0 = \alpha + 2\alpha_i.$$

This completes the proof. □

Using this Lemma we find the normal form for palindromes in $\mathrm{N}_{n,2}$.

Proposition 2.7. *There are n different types of palindromes in $\mathrm{N}_{n,2}$ that can be written in the following normal forms.*

$$p_1 = x_1^{\alpha_1} x_2^{2\alpha_2} \dots x_n^{2\alpha_n} \prod_{t=2}^{n} z_{t1}^{\alpha_t\alpha_1} \prod_{2\le l<k\le n} z_{kl}^{2\alpha_k\alpha_l},$$

$$\vdots$$

$$p_j = x_1^{2\alpha_1} \dots x_{j-1}^{2\alpha_{j-1}} x_j^{\alpha_j} x_{j+1}^{2\alpha_{j+1}} \dots x_n^{2\alpha_n} \prod_{\substack{1\le l<k\le n\\ k,l\neq j}} z_{kl}^{2\alpha_k\alpha_l} \cdot \prod_{t=1}^{j-1} z_{jt}^{\alpha_j\alpha_t} \cdot \prod_{s=j+1}^{n} z_{sj}^{\alpha_s\alpha_j},\ 2\le j\le n-1.$$

$$\vdots$$

$$p_n = x_1^{2\alpha_1} \dots x_{n-1}^{2\alpha_{n-1}} x_n^{\alpha_n} \prod_{1\le l<k\le n-1} z_{kl}^{2\alpha_l\alpha_k} \cdot \prod_{t=1}^{n-1} z_{nt}^{\alpha_n\alpha_t}.$$

Proof. We give a proof for the case p_j, $2\le j\le n-1$. The cases p_1 and p_n are similar and simpler. By Lemma 2.6 we have

$$p_j = x_1^{\alpha_1} \dots x_{j-1}^{\alpha_{j-1}} x_{j+1}^{\alpha_{j+1}} \dots x_n^{\alpha_n} \cdot x_j^{\alpha_j} \cdot x_n^{\alpha_n} \dots x_{j+1}^{\alpha_{j+1}} x_{j-1}^{\alpha_{j-1}} \dots x_1^{\alpha_1}.$$

Remove the element $x_j^{\alpha_j}$ to the left

$$p_j = x_1^{\alpha_1} \dots x_{j-1}^{\alpha_{j-1}} x_j^{\alpha_j} x_{j+1}^{\alpha_{j+1}} \dots x_n^{2\alpha_n} \dots x_{j+1}^{\alpha_{j+1}} x_{j-1}^{\alpha_{j-1}} \dots x_1^{\alpha_1} \cdot z_{nj}^{\alpha_n\alpha_j} z_{n-1,j}^{\alpha_{n-1}\alpha_j} \dots z_{j+1,j}^{\alpha_{j+1}\alpha_j}.$$

Removing the right occurrence of $x_1^{\alpha_1}$ to the left

$$p_j = x_1^{2\alpha_1} \dots x_{j-1}^{\alpha_{j-1}} x_j^{\alpha_j} x_{j+1}^{\alpha_{j+1}} \dots x_n^{2\alpha_n} \dots x_{j+1}^{\alpha_{j+1}} x_{j-1}^{\alpha_{j-1}} \dots x_2^{\alpha_2} \cdot \prod_{t=j+1}^{n} z_{tj}^{\alpha_t\alpha_j} \cdot z_{j1}^{\alpha_j\alpha_1} \cdot \prod_{\substack{t=2\\ t\neq j}}^{n} z_{t1}^{2\alpha_t\alpha_1}.$$

By the similar manner removing the right occurrence of $x_2^{\alpha_2}, \dots, x_{j-1}^{\alpha_{j-1}}$ to the left we get

$$p_j = x_1^{2\alpha_1} \dots x_{j-1}^{2\alpha_{j-1}} x_j^{\alpha_j} \left(x_{j+1}^{\alpha_{j+1}} \dots x_n^{2\alpha_n} \dots x_{j+1}^{\alpha_{j+1}} \right) c$$

where

$$c = \prod_{t=j+1}^{n} z_{tj}^{\alpha_t\alpha_j} \cdot \prod_{l=1}^{j-1} z_{jl}^{\alpha_j\alpha_l} \cdot \prod_{\substack{t=2\\ t\neq j}}^{n} z_{t1}^{2\alpha_t\alpha_1} \prod_{\substack{t=3\\ t\neq j}}^{n} z_{t2}^{2\alpha_t\alpha_2} \dots \prod_{t=j+1}^{n} z_{t,j-1}^{2\alpha_t\alpha_{j-1}}.$$

Represent the expression in the brackets in the normal form

$$x_{j+1}^{\alpha_{j+1}} \dots x_n^{2\alpha_n} \dots x_{j+1}^{\alpha_{j+1}} = x_{j+1}^{2\alpha_{j+1}} x_{j+2}^{2\alpha_{j+2}} \dots x_n^{2\alpha_n} \cdot \prod_{j+1\le l<k\le n} z_{kl}^{2\alpha_k\alpha_l}.$$

Hence

$$p_j = x_1^{2\alpha_1} \dots x_{j-1}^{2\alpha_{j-1}} x_j^{\alpha_j} x_{j+1}^{2\alpha_{j+1}} \dots x_n^{2\alpha_n} \cdot \prod_{t=j+1}^{n} z_{tj}^{\alpha_t\alpha_j} \cdot \prod_{l=1}^{j-1} z_{jl}^{\alpha_j\alpha_l} \cdot \prod_{\substack{1\le l\le j-1\\ l<k\le n\\ k\neq j}} z_{kl}^{2\alpha_k\alpha_l} \cdot \prod_{j+1\le l<k\le n} z_{kl}^{2\alpha_k\alpha_l}.$$

We see that this expression is equal to the needed formula. ☐

3. Palindromic Width of Some Abelian-By-Nilpotent Groups

3.1. **Palindromic Width of a Metabelian Group.** In [3], we proved that if AN_n is a free abelian-by-nilpotent group with basis $X = \{x_1, \dots, x_n\}$, then $n \leq \mathrm{pw}(AN_n, X) \leq 5n$. To prove this we used the following representation of elements of AN_n that follows from [2, Theorem 2].

Theorem 3.1. *Let $AN_n = \langle x_1, \dots, x_n \rangle$ be a non-abelian free abelian-by-nilpotent group of rank n. Let A be an abelian normal subgroup of AN_n such that AN_n/A is nilpotent. Then every element $g \in AN_n$ can be expressed as:*

$$g = x_1^{\alpha_1} \dots x_n^{\alpha_n} [u_1, x_1]^{a_1} [u_2, x_2]^{a_2} \dots [u_n, x_n]^{a_n}$$

for $u_1, \dots, u_n \in AN_n$ and $a_1, \dots, a_n \in A$, $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$.

Evidently, every metabelian group is an abelian-by-nilpotent group. However, for finitely generated metabelian groups, we provide a better upper bound.

Theorem 3.2. *Let M_n be a free metabelian group of rank n. Then*

$$\mathrm{pw}(M_n, X) \leq 4n - 1.$$

Proof. Let $h \in \gamma_2(M_n)$. Using the fact that $\gamma_2(M_n)$ is abelian, we let in Theorem 3.1 $A = \gamma_2(M_n)$. Hence h has the form

$$h = [u_1, x_1] \dots [u_n, x_n] \quad \text{for } u_1, \dots, u_n \in \gamma_2(F_n(\mathcal{U}^2)).$$

Hence every $g \in M_n$ has the form:

$$g = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} [u_1, x_1] \dots [u_n, x_n].$$

Observe that

$$x_n^{\alpha_n} [u_n, x_n] = [u_n, x_n]^{x_n^{-\alpha_n}} x_n^{\alpha_n} = [v_n, x_n] x_n^{\alpha_n}$$

for $v_n = x_n^{\alpha_n} u_n x_n^{-\alpha_n}$. So, g can be written as

$$g = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_{n-1}^{\alpha_{n-1}} [v_n, x_n] x_n^{\alpha_n} [u_1, x_1] \dots [u_{n-1}, x_{n-1}].$$

Now it follows from Lemma 2.2(3) that

$$\mathrm{pw}(M_n, X) \leq (n-1) + 3 + 3(n-1) = 4n - 1.$$

This proves the result. □

3.2. **Palindromic Width of $\widetilde{AN}_n$.**

Lemma 3.3. *Let G be a group which is generated by the set of involutions $Y = \{y_1, y_2, \dots, y_n\}$. Let g, h be any element in G and p be a palindrome in G. Then the following hold*

1. *Any commutator of the type $[g, p]$ is a product of 2 palindromes. Any element $[g, y]y^{\alpha}$, $y \in Y$, $\alpha \in \{0, 1\}$ is a product of 2 palindromes.*
2. *Any commutator of the type $[g, y]^h$ is a product of 2 palindromes.*

Proof. (1) See that

$$[g, p] = g^{-1} p^{-1} g p = \overline{g}\, \overline{p}\, g \cdot p$$

is a product of palindromes $\overline{g}\, \overline{p}\, g$ and p.

Similarly,

$$[g, y] y^{\alpha} = \overline{g} y g y^{1+\alpha},$$

which is a palindrome if $\alpha = 1$ or a product of two palindromes if $\alpha = 0$.

(2) We have

$$[g, y]^h = h^{-1}g^{-1}ygyh = \overline{h}\overline{g}ygh \cdot \overline{h}yh$$

is a product of 2 palindromes. □

3.2.1. *Proof of Theorem 1.1.*

Proof. (i) We have $\widetilde{\mathrm{N}}_{n,1} = \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ (n-times). We see that there is a homomorphism $\widetilde{AN}_n \to \widetilde{\mathrm{N}}_{n,1}$. Now the left-side of the inequality follows from the fact $\mathrm{pw}(\widetilde{\mathrm{N}}_{n,1}) = n$, see [3] for a proof of this fact. To prove the right-hand side inequality, write any element $g \in \widetilde{AN}_n$ in the form

$$g = y_1^{\epsilon_1} \ldots y_n^{\epsilon_n}[g_1, y_1]^{a_1}[g_2, y_2]^{a_2} \ldots [g_n, y_n]^{a_n},$$

where for $i = 1, \ldots, n$, $\epsilon_i \in \{0, 1\}$, $g_i \in \widetilde{AN}_n$ and $a_i \in \widetilde{A}$; here $\widetilde{A}$ is the image of A under the homomorphism $AN_n \to \widetilde{AN}_n$. Such a representation of g follows from Theorem 3.1. By Lemma 3.3, for $i = 1, \ldots, n$, $[g_i, y_i]^{a_i}$ is a product of 2 palindromes. Hence g is a product of at most $n + 2n = 3n$ palindromes. This proves the first part of the theorem.

(ii) Using the fact that $\widetilde{M}'_n$ is abelian, we set in Theorem 3.1, $A = \widetilde{M}'_n$. Since any two commutators of $\widetilde{M}_n$ commute, hence every $g \in \widetilde{M}_n$ has the form:

$$g = y_1^{\varepsilon_1}y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n}[u_1, y_1] \ldots [u_n, y_n] = y_1^{\varepsilon_1}y_2^{\varepsilon_2} \ldots y_n^{\varepsilon_n}[u_n, y_n][u_1, y_1] \ldots [u_{n-1}, y_{n-1}]$$

for some $\epsilon_i \in \{0, 1\}$, $u_i \in \widetilde{M}_n$. Observe that

$$y_n^{\varepsilon_n}[u_n, y_n] = [u_n, y_n]^{y_n^{-\varepsilon_n}}y_n^{\varepsilon_n} = [v_n, y_n]y_n^{\varepsilon_n}$$

where $v_n = y_n^{\varepsilon_n}u_ny_n^{-\varepsilon_n}$. So, g can be written as

$$g = y_1^{\varepsilon_1}y_2^{\varepsilon_2} \ldots y_{n-1}^{\varepsilon_{n-1}}[v_n, y_n]y_n^{\varepsilon_n}[u_1, y_1] \ldots [u_{n-1}, y_{n-1}].$$

Now it follows from Lemma 3.3 that

$$\mathrm{pw}(\widetilde{M}_n, Y) \leq n - 1 + 2 + 2(n - 1) = 3n - 1.$$

This proves the result. □

4. Proof of Theorem 1.2

Now consider the group $\widetilde{\mathrm{N}}_{n,r} = \mathrm{N}_{n,r}/\langle\langle x_1^2, \cdots, x_n^2\rangle\rangle$. Let $z_{ij} = [y_i, y_j]$, $1 \leq j < i \leq n$. For $i = 1, \ldots, n$, $y_i = y_i^{-1}$ in $\widetilde{\mathrm{N}}_{n,r}$.

In this Section we prove:.

Theorem 1.2. (1) *For $n, r \geq 2$, $n \leq \mathrm{pw}(\widetilde{\mathrm{N}}_{n,r}) \leq 2n$.*
(2) *The palindromic width of $\widetilde{\mathrm{N}}_{n,2}$ is $2(n - 1)$.*

The first part of this Theorem follows from the following assertion.

Lemma 4.1. *For $n, r \geq 2$, $n \leq \mathrm{pw}(\widetilde{\mathrm{N}}_{n,r}) \leq 2n$.*

Proof. We proved in [3] that $\mathrm{pw}(\widetilde{\mathrm{N}}_{n,1}) = n$. Hence, the left hand side inequality holds.

We claim that any element g in $\widetilde{\mathrm{N}}_{n,r}$, $r \geq 2$, can be represented in the form

$$g = [u_1, y_1]y_1^{\alpha_1}[u_2, y_2]y_2^{\alpha_2} \ldots [u_n, y_n]\, y_n^{\alpha_n},\ \alpha_i \in \{0, 1\}.$$

We shall use induction on r. If $r = 2$ and $g \in \widetilde{\mathrm{N}}_{n,2}$ then it follows from Lemma 2.4 that

$$g = y_1^{\alpha_1}\, y_2^{\alpha_2}\, \ldots\, y_n^{\alpha_n}\, [u_1, y_1]\, [u_2, y_2]\, \ldots\, [u_n, y_n],\ \alpha_i \in \{0, 1\},\ u_i \in \widetilde{\mathrm{N}}_{n,2}.$$

But the commutators $[u_i, y_i]$, $i = 1, 2, \dots, n$, lie in the center of $\widetilde{\mathrm{N}}_{n,2}$. Hence

$$g = [u_1, y_1]\, y_1^{\alpha_1}\, [u_2, y_2]\, y_2^{\alpha_2}\, \dots\, [u_n, y_n]\, y_n^{\alpha_n},\ \alpha_i \in \{0, 1\}.$$

has the required form. Let the result holds for groups $\widetilde{\mathrm{N}}_{n,r}$. We claim that the result also holds for $\widetilde{\mathrm{N}}_{n,r+1}$. Let $\Gamma = \gamma_{r+1}(\widetilde{\mathrm{N}}_{n,r+1}) = [\gamma_r(\widetilde{\mathrm{N}}_{n,r+1}), \widetilde{\mathrm{N}}_{n,r+1}]$. Then an element g of $\widetilde{\mathrm{N}}_{n,r+1}$ has the form

$$g = [u_1, y_1]\, y_1^{\alpha_1}\, [u_2, y_2]\, y_2^{\alpha_2}\, \dots\, [u_n, y_n]\, y_n^{\alpha_n}\, d$$

for some $d \in \Gamma$, $\alpha_i \in \{0, 1\}$. It follows from Lemma 2.3,

$$d = [a_1, y_1]\, [a_2, y_2]\, \dots\, [a_n, y_n],\ \text{for some } a_i \in \gamma_r(\widetilde{\mathrm{N}}_{n,r+1}).$$

Since all $[a_i, y_i]$ lie in the center of $\widetilde{\mathrm{N}}_{n,r+1}$, hence

$$g = [u_1, y_1]\, [a_1, y_1]\, y_1^{\alpha_1}\, [u_2, y_2]\, [a_2, y_2]\, y_2^{\alpha_2}\, \dots\, [u_n, y_n]\, [a_1, y_1]\, y_n^{\alpha_n} =$$
$$= [u_1 a_1, y_1]\, y_1^{\alpha_1}\, [u_2 a_2, y_2]\, y_2^{\alpha_2}\, \dots\, [u_n a_n, y_n]\, y_n^{\alpha_n}$$

has the required form.

By Lemma 3.3, any element $[u_i, y_i]\, y_i^{\alpha_i}$ is a product of 2 palindromes and g is a product of $2n$ palindromes. □

The following lemma follows by imitating the proof of [3, Lemma 3.6] and using Lemma 3.3.

Lemma 4.2. *Any element in $\widetilde{\mathrm{N}}_{n,2}$, $n \geq 2$ is a product of at most $2(n-1)$ palindromes.*

To prove that the palindromic width of $\widetilde{\mathrm{N}}_{n,2}$ is at least $2(n-1)$, it is enough to find some element in $\widetilde{\mathrm{N}}_{n,2}$ that can not be represented as a product of less than $2(n-1)$ palindromes. To do this we introduce some notations. Let

$$Bas_n = \{z_{kl} \mid 1 \leq l < k \leq n\}$$

is the set of all basis commutators of weight 2 in $\widetilde{\mathrm{N}}_{n,2}$. Any palindrome in $\widetilde{\mathrm{N}}_{n,2}$ has a normal form that is the image of the normal forms of palindromes of $\mathrm{N}_{n,2}$ obtained in Proposition 2.7. We shall use the same symbol $p_1, p_2, \dots, p_n$ to denote the normal forms in $\widetilde{\mathrm{N}}_{n,2}$. We have

$$p_1 = y_1^{\alpha_1}\, z_{21}^{\alpha_1\alpha_2}\, z_{31}^{\alpha_1\alpha_3} \dots z_{n1}^{\alpha_1\alpha_n},$$
$$\vdots$$
$$p_j = y_j^{\alpha_j} \cdot \prod_{t=1}^{j-1} z_{jt}^{\alpha_j\alpha_t} \cdot \prod_{s=j+1}^{n} z_{sj}^{\alpha_s\alpha_j},\ 2 \leq j \leq n-1,$$
$$\vdots$$
$$p_n = y_n^{\alpha_n} \cdot \prod_{t=1}^{n-1} z_{nt}^{\alpha_n\alpha_t}.$$

If $w \in \widetilde{\mathrm{N}}_{n,2}$ is some element that is represented in the normal form, then denote by $b(w)$ the set of basis commutators of weight 2 those are in this normal form. For example

$$b(p_1) = \{z_{21}, z_{31}, \dots, z_{n1}\},$$
$$\vdots$$
$$b(p_j) = \{z_{j1}, \dots, z_{j,j-1}, z_{nj}, \dots, z_{j+1,j}\},\ 2 \leq j \leq n-1.$$
$$\vdots$$

$$b(p_n) = \{z_{n1}, \dots, z_{n,n-1}\}.$$

If $w_1, \dots, w_k \in \widetilde{\mathrm{N}}_{n,2}$ are represented in the normal form, then denote

$$b(w_1, \dots, w_k) = \bigcup_{i=1}^{k} b(w_i).$$

Lemma 4.3. (1) *$b(p_1, \dots, p_n) = Bas_n$.*

(2) *For arbitrary i, $1 \le i \le n$, $b(p_1, \dots, p_n) - b(p_i) = Bas_n$.*

(3) *$b(p_1, \dots, p_n) - (b(p_i) \cup b(p_j)) \ne Bas_n$ if $1 \le i < j \le n$.*

Proof. (1) follows from the fact that any basic commutator $z_{kl},\ 1 \le l < k \le n$ appeared in p_j, $1 \le j \le n$.

(2) Note that any commutator z_{kl} is contained in the normal forms p_k and p_l. Hence if we remove $b(p_i)$ from $b(p_1, \dots, p_n)$, then we will have all commutators of Bas_n.

(3) Note that $b(p_1, \dots, p_n) - (b(p_i) \cup b(p_j))$ does not contain z_{ji}. □

Lemma 4.4. *The element $g = \prod_{1 \le l < k \le n} z_{kl}$ in $\widetilde{\mathrm{N}}_{n,2}$ can not be written as a product of less than $2(n-1)$ palindromes.*

Proof. We see that $b(g) = Bas_n$. Hence to represent g as a product of palindromes, we must take at least $n-1$ different types of palindromes. Suppose that $g = q_1 \dots q_s,\ n-1 \le s < 2(n-1)$ is a product of s palindromes. Since at least $n-1$ types of palindromes are included in this product and $s < 2(n-1)$, there is a palindrome of some type that appears only one time in the product. Without loss of generality, we can assume that it is the palindrome of type p_1, say,

$$q_1 = y_1 c,\ c \in \widetilde{\mathrm{N}}'_{n,2}.$$

Then the product $g = q_1, \dots, q_s$ does not lie in the commutator subgroup $\widetilde{\mathrm{N}}'_{n,2}$, since in the normal form it contains element y_1. This is a contradiction. Hence g can not be written as a product of less that $2(n-1)$ palindromes. □

4.0.2. *Proof of the second part of Theorem 1.2.*

Proof. We have already proved earlier that $\mathrm{pw}(\widetilde{\mathrm{N}}_{n,2}) \le 2(n-1)$. It follows from Lemma 4.4 that there exists at least one element in $\widetilde{\mathrm{N}}_{n,2}$ whose palindromic length is at least $2(n-1)$. Thus $\mathrm{pw}(\widetilde{\mathrm{N}}_{n,2}) \ge 2(n-1)$. □

Proof of Corollary 1.

Proof. There is an onto homomorphism from $\mathrm{N}_{n,2}$ onto $\widetilde{\mathrm{N}}_{n,2}$. Hence $\mathrm{pw}(\widetilde{\mathrm{N}}_{n,2}) \le \mathrm{pw}(\mathrm{N}_{n,2})$ by Lemma 2.1. The corollary now follows from Lemma 2.5 and [3, Theorem 1.1]. □

5. Proof of Proposition 1.3

It follows from Proposition 2.7 that palindromes in $\widetilde{\mathrm{N}}_{3,2}$ are of the following form:

$$p_{(\alpha_0, 2\alpha_1, 2\alpha_2)} = y_1^{\alpha_0} z_{21}^{\alpha_0\alpha_1} z_{31}^{\alpha_0\alpha_2}, \tag{5.1}$$

$$p_{(2\alpha_1, \alpha_0, 2\alpha_2)} = y_2^{\alpha_0} z_{21}^{\alpha_0\alpha_1} z_{32}^{\alpha_0\alpha_2}, \tag{5.2}$$

$$p_{(2\alpha_1, 2\alpha_2, \alpha_0)} = y_3^{\alpha_0} z_{31}^{\alpha_0\alpha_1} z_{32}^{\alpha_0\alpha_2}. \tag{5.3}$$

In the following, for simplicity, we denote the palindromes of the form (5.1), (5.2) and (5.3) by p_1, p_2 and p_3 respectively forgetting the subscripts. When we write a product, for eg. $p_1p_1p_1$, it should be understood that each p_1 is a palindrome of the type (5.1) but not necessarily with the same subscript unless it is mentioned otherwise. The rest of this Section will be devoted to the proof of Proposition 1.3.

From Lemma 4.4 follows lemma.

Lemma 5.1. *For $1 \leq j < i \leq 3$ let $z_{ij} = [y_i, y_j]$ in $\widetilde{\mathrm{N}}_{3,2}$. The element $g = z_{21}z_{31}z_{32}$ in $\widetilde{\mathrm{N}}_{3,2}$ has palindromic length $l_{\mathcal{P}}(g)$ is equal 4.*

Proof. From Lemma 4.4 follows that $l_{\mathcal{P}}(g) \geq 4$.

On the other hand, note that

$$\begin{aligned} g &= [y_2, y_1][y_3, y_1][y_3, y_2] \\ &= [y_3, y_2][y_2, y_1][y_3, y_1] \\ &= y_2[y_3, y_2]y_2[y_2, y_1]y_1[y_3, y_1]y_1 \\ &= p_2p_2p_1p_1 \end{aligned}$$

Thus g can be expressed as a product of four palindromes. □

Note that any element w of $\widetilde{\mathrm{N}}_{3,2}$ has the form

$$w = y_1^{a_1}\, y_2^{a_2}\, y_3^{a_3} z_{21}^{b_1} z_{31}^{b_2} z_{32}^{b_3},$$

where, for $i = 1, 2, 3$, a_i, $b_i \in \{0, 1\}$. Define

$$|w| = \sum_{i=1}^{3}(a_i + b_i).$$

If $|w| = 1$ then $l_{\mathcal{P}}(w) \leq 2$, since, any commutator z_{ij} is a product of two palindromes.

Let $|w| = 2$, then we have 15 possibilities for $(a_1, a_2, a_3, b_1, b_2, b_3)$, where each of the a_i and b_i is either 0 or 1. For simplicity of notation we identify the 6-tuple $(a_1, a_2, a_3, b_1, b_2, b_3)$ with the binary word $a_1a_2a_3b_1b_2b_3$ and write down the 15 possibilities below:

110000, 101000, 100100, 100010, 100001, 011000, 010100, 010010, 010001, 001100, 001010, 001001, 000110, 000101, 000011.

In the first twelve cases we have a product of two generators or a product of one generator and a commutator. The palindromic length of this product is ≤ 3. In the last three cases we have:

$$000110: \ w = z_{21}z_{31} = y_2y_1y_2 \cdot y_3y_1y_3 \cdot y_1.$$
$$000101: \ w = z_{21}z_{32} = z_{32}z_{21} = y_3y_2y_3 \cdot y_1y_2y_1.$$
$$000011: \ w = z_{31}z_{32} = y_3y_1y_3y_1y_3 \cdot y_2y_3y_2.$$

Thus in each cases w is a product of at most three palindromes.

Let $|w| = 3$, then we have $\binom{6}{3} = 20$ possibilities:

111000, 110100, 110010, 110001, 101100, 101010, 101001, 100110 100101, 100011, 011100, 011010, 011001, 010110, 010101, 010011, 001110, 001101, 001011, 000111.

After rearranging terms and simplification we get:

$$110010: \ w = y_1y_2z_{31} = z_{31}y_1y_2 = y_3y_2y_3 \cdot y_2.$$

$$110001: \ w = y_1y_2z_{32} = y_1z_{32}y_2 = y_1 \cdot y_3y_2y_3; \quad 101100: \ w = z_{21}y_1y_3 = y_2y_1y_2 \cdot y_3.$$

$$101010: \ w = y_1y_3z_{31} = y_3y_1; \quad 101001: \ w = y_1y_3z_{32} = y_1 \cdot y_2y_3y_2.$$

$$100110: \ w = y_1z_{21}z_{31} = z_{21}y_1z_{31} = y_2y_1y_2y_1 \cdot y_1 \cdot y_3y_1y_3y_1 = y_2y_1y_2 \cdot y_3y_1y_3 \cdot y_1.$$

$$100101: \ w = y_1z_{21}z_{32} = z_{32}z_{21}y_1 = y_3y_2y_3 \cdot y_1 \cdot y_2.$$

$$100011: \ w = y_1z_{31}z_{32} = z_{31}y_1z_{32} = y_3y_1 \cdot y_2y_3y_2.$$

$$010110: \ w = y_2z_{21}z_{31} = y_1y_2y_1 \cdot y_3y_1y_3 \cdot y_1. \ \ 010101: \ w = y_2z_{21}z_{32} = y_1y_2y_1 \cdot y_3y_2y_3 \cdot y_2.$$

$$010011: \ w = z_{31}z_{32}y_2 = y_3y_1y_3 \cdot y_1 \cdot y_3y_2y_3. \quad 001110: \ w = z_{21}y_3z_{31} = y_2y_1y_2 \cdot y_3y_1.$$

$$001101: \ w = z_{21}y_3z_{32} = y_2y_1y_2 \cdot y_1 \cdot y_2y_3y_2. \ \ 001011: \ w = y_3z_{31}z_{32} = y_1y_3y_1 \cdot y_3y_2y_3 \cdot y_2.$$

Thus we see that in each of the above cases, w is a product of at most three palindromes. Finally $000111: \ w = z_{21}z_{31}z_{32}$ is a product of four palindromes as we have seen in Lemma 5.1.

Let $|w| = 4$. Then we have $\binom{6}{4} = 15$ possibilities:

111100, 111010, 110110, 101110, 011110, 111001, 110101, 101101, 011101, 110011, 101011, 011011, 100111, 010111, 001111.

We have after rearranging terms and simplification,

$$111100: \ w = y_1y_2y_3z_{21} = y_1y_2z_{21}y_3 = y_1y_2y_2y_1y_2y_1y_3 = y_2y_1y_3.$$

$$110110: \ w = y_1z_{21}z_{31} = z_{31}y_1y_2z_{21} = y_3y_1y_3 \cdot y_1y_2y_1.$$

$$101110: \ w = y_1y_3z_{21}z_{31} = z_{21}y_1y_3z_{31} = y_2y_1y_2 \cdot y_1y_3y_1.$$

$$011110: \ w = y_2z_{21}y_3z_{31} = y_1(y_2y_3)y_1; \quad 111001: \ w = y_1y_2y_3z_{32} = y_1y_3y_2.$$

$$110101: \ w = y_1y_2z_{21}z_{32} = y_2y_1 \cdot y_3y_2y_3y_2 = y_2y_1y_2 \cdot y_2y_3y_2y_3y_2.$$

$$101101: \ w = z_{21}y_1y_3z_{32} = y_2y_1y_2 \cdot y_2y_3y_2.$$

$$011101: \ w = y_2z_{21}y_3z_{32} = y_1y_2y_1 \cdot y_2y_3y_2; \ \ 110011: \ w = z_{31}y_1z_{32}y_2 = y_3y_1y_2y_3 = y_3y_1y_3 \cdot y_3y_2y_3.$$

$$101011: \ w = z_{31}y_1y_3z_{32} = y_3y_1y_3 \cdot y_2y_3y_2;$$

$$011011: \ w = z_{31}y_1z_{32}z_{21} = y_3y_1y_3 \cdot y_3y_2y_3 \cdot y_1y_2y_1.$$

$$010111: \ w = y_2z_{21}z_{31}z_{32} = y_1y_2y_1 \cdot y_3y_1y_3y_1y_3 \cdot y_2y_3y_2.$$

$$001111: \ w = y_3z_{21}z_{31}z_{32} = z_{21}y_3z_{31}z_{32} = y_2y_1y_2 \cdot y_3y_1y_3 \cdot y_2y_3y_2.$$

Thus we see that in each of the above cases w is a product of at most three palindromes.

Let $|w| = 5$. There are six possibilities and after rearranging terms and simplification we have:

$$111110: \ w = y_1y_2y_3z_{21}z_{31} = y_1y_2z_{21}y_3z_{31} = y_2y_3y_1.$$

$$111101: \ w = y_1y_2y_3z_{21}z_{32} = y_1y_2z_{21}y_3z_{32} = y_2y_1y_2 \cdot y_3y_2.$$

$$111011: \ w = y_1y_2y_3z_{31}z_{32} = y_1z_{31}y_2y_3z_{32} = y_1y_3y_1 \cdot y_3y_1y_3 \cdot y_2.$$

$$110111: \ w = y_1y_2z_{21}z_{31}z_{32} = y_2y_1z_{31}z_{32} = z_{32}y_2y_1z_{31} = y_3y_2y_3 \cdot y_1y_3y_1y_3y_1.$$

$$101111: \ w = y_1y_3z_{21}z_{31}z_{32} = y_1y_3z_{31}z_{32}z_{21} = y_3y_1z_{32}z_{21} = y_3z_{32}z_{21}y_1 = y_2y_3y_2 \cdot y_2y_1y_2.$$

$$011111: \ w = y_2y_3z_{21}z_{31}z_{32} = z_{21}z_{31}y_2y_3z_{32} = y_3y_1x_3x_1y_3 \cdot y_1y_2y_1.$$

Thus w is a product of at most three palindromes.

Let $|g| = 6$. Then the only possibility is 111111 and we have

$$w = y_1y_2y_3z_{21}z_{31}z_{32} = y_1y_2z_{21}y_3z_{31}z_{32} = y_2 \cdot y_3y_1y_3 \cdot y_2y_3y_2.$$

Thus we have shown that all but $g = z_{21}z_{31}z_{32}$ in $\widetilde{\mathrm{N}}_{3,2}$ can be written as a product of at most three palindromes. From Lemma 5.1 it follows that the element g is the only element whose palindromic length is 4. This proves Proposition 1.3.

References

[1] Kh. S. Allambergenov, V. A. Roman'kov, *On products of commutators in groups*, (Russian) Depon. VINITI, 1985, no. 4566-85, 20 pp.

[2] M. Akhavan-Malayeri, A. Rhemtulla, *Commutator length of abelian-by-nilpotent groups*. Glasg. Math. J. 40 (1998), no.1, 117–121.

[3] V. G. Bardakov and K. Gongopadhyay, *Palindromic width of free nilpotent groups*, J. Algebra 402 (2014), 379–391.

[4] V. Bardakov, V. Shpilrain, V. Tolstykh, *On the palindromic and primitive widths of a free group*, J. Algebra, 285 (2005), 574–585.

[5] V. Bardakov, V. Tolstykh, *The palindromic width of a free product of groups*, J. Aust. Math. Soc., 81 (2006), no. 2, 199–208.

[6] D. Carter and G. Keller, *Bounded elementary generation of* $\mathrm{SL}_n(\mathcal{O})$, Amer. J. Math 105, no. 3 (1983), 673–687.

[7] D. Collins, *Palindromic automorphism of free groups*, in: Combinatorial and Geometric Group Theory, London Math. Soc. Lecture Note Ser., vol. 204, Cambridge Univ. Press, Cambridge, 1995, 63 – 72.

[8] H. Glover, C. Jensen, *Geometry for palindromic automorphisms of free groups*, Comment. Math. Helv. 75 (2000), 644–667.

[9] J. Gilman, L. Keen, *Enumerating palindromes and primitives in rank two free groups*, J. Algebra 332 (2011), 1–13.

[10] J. Gilman, L. Keen, *Discreteness criteria and the hyperbolic geometry of palindromes*, Conform. Geom. Dyn. 13 (2009), 76–90.

[11] H. Helling, *A note on the automorphism group of the rank two free group*, J. Algebra 223 (2000), 610–614.

[12] W. Magnus, A. Karrass, D. Solitar, Combinatorial group theory: Presentations of Groups in Terms of Generators and Relations, Interscience Publishers, New York, 1996.

[13] A. Piggott, *Palindromic primitives and palindromic bases in the free group of rank two*, J. Algebra 304 (2006), 359–366.

[14] A. H. Rhemtulla, *Commutators of certain finitely generated soluble groups*, Can. J. Math. 21 (1969), 1160–1164.

[15] T. R. Riley and A. W. Sale, *Palindromic width of metabelian groups*, arXiv:1307.4861.

[16] B. Sury, *Bounded generation does not imply finite presentation*, Comm. Algebra 25, no. 5 (1997), 1673–1683.

Sobolev Institute of Mathematics, Novosibirsk State University, Novisibirsk 630090

and

Laboratory of Quantum Topology, Chelyabinsk State University, BratevKashirinykh street 129, Chelyabinsk 454001, Russia

E-mail address: `bardakov@math.nsc.ru`

Department of Mathematical Sciences, Indian Institute of Science Education and Research (IISER) Mohali, Knowledge City, Sector 81, S.A.S. Nagar, P.O. Manauli 140306, India

E-mail address: `krishnendu@iisermohali.ac.in, krishnendug@gmail.com`