Infinite dimensional topological field theories from Hurwitz numbers

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ABSTRACT

Classical Hurwitz numbers of a fixed degree together with Hurwitz numbers of seamed surfaces give rise to a Klein topological field theory (see [4]). We extend this construction to Hurwitz numbers of all degrees simultaneously. The corresponding infinite dimensional Cardy-Frobenius algebra is computed in terms of Young diagrams and bipartite graphs. This algebra turns out to be isomorphic to the algebra of differential operators introduced in [19, 20], which serves a model for open-closed string theory. We prove that the operators corresponding to Young diagrams and bipartite graphs give rise to relations between Hurwitz numbers.

Contents

1. Introduction

According to [4], classical Hurwitz numbers of a fixed degree n together with Hurwitz numbers of seamed surfaces[1] of degree n give rise to an open-closed topological field theory in dimension 2. The corresponding Cardy-Frobenius algebra includes the algebra of Young diagrams of degree n see [7], the algebra of bipartite graphs of degree n see [3] and [4]. This Cardy-Frobenius algebra can be considered as a realization [20] of an open-closed string theory in the sense of [15], [13], [16].

In this paper, we extend this construction to the algebra of Hurwitz numbers of all degrees simultaneously. We prove that this universal Hurwitz algebra forms an infinite-dimensional topological field theory, and is isomorphic to the Cardy-Frobenius algebra

[1]The seamed surfaces were accurately defined in [24]. They are also called world-sheet foams in [12].

of differential operators constructed in [19]. The latter algebra is generated by the differential operators associated with arbitrary Young diagrams and bipartite graphs. The operator associated with the Young diagram of a transposition coincides with the cut-and-join operator generating a differential relation for the generating function of the special Hurwitz numbers [8]. The operators associated with the simplest bipartite graphs are found in [22]. We prove that all the operators associated with Young diagrams and bipartite graphs give rise to analogous cut-and-join differential relations. Most of these results are extended to the case of non-orientable surfaces.

Acknowledgements. S.N. is grateful to MPIM and IHES for the kind hospitality and support. Our work is partly supported by Ministry of Education. The work of third author (S.N.) was partially supported by Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020). Also, our work is partly supported by NSh-1500.2014.2 (A.Mir. and A.Mor.) and NSh-5138.2014.1 (S.N.), by RFBR grants 13-02-00457 (A.Mir.), 13-02-00478(A.Mor.), 13-01-00755(S.N.), by joint grants 13-02-91371-ST, 14-01-92691-Ind, by the Brazil National Counsel of Scientific and Technological Development (A.Mor.). This study was carried out within "The National Research University Higher School of Economics' Academic Fund Program in 2013-2014, research grant No.12-01-0122" (S.N.).

2. Hurwitz numbers of seamed surfaces

2.1. **Seamed surfaces.** We shall consider *seamed graphs*, that is one dimensional spaces consisting of finitely many vertices and edges, connecting vertices. Note that an edge may form a loop, that is may connect a vertex to itself.

A *seamed surface* Ω is a triple (S, Δ, φ), where

- $\Delta = \Delta(\Omega)$ is a seamed graph;
- $S = S(\Omega)$ is a compact 2-manifold, (possibly non-connected and non-orientable) such that every connected components of S has non-empty boundary, if $\Delta \neq \emptyset$;
- $\varphi = \varphi_\Omega : \partial S \to \Delta$ is *the gluing map*, that is, a map such that:
  a) $\operatorname{Im} \varphi = \Delta$;
  b) on each connected component of $c \subset \partial S$, φ is a covering of $\varphi(c)$;
  c) the below condition $(*)$ is fulfilled.

The result of the gluing of S along φ (i.e. the space $S \cup \Delta$ with x and $\varphi(x)$ identified for all $x \in \partial S$) will be denoted by Ω_φ.

(*) the punctured neighborhood of every vertex of the graph Δ in Ω_φ is connected. Here by a punctured neighborhood of a point x we mean a neighborhood with the point x deleted.

Example 2.1. *Consider a compact 2-manifold, such that every connected component of S has non-empty boundary. Fix on ∂S a finite set of points, transforming ∂S to a seamed graph Δ. Then $\Omega = (S, \Delta, id : \partial S \to \Delta)$ be a seamed surface.*

An *isomorphism of seamed surfaces* (S, Δ, φ) and (S', Δ', φ') is a homeomorphism $f : \Omega_\varphi \to \Omega'_{\varphi'}$ generating an isomorphism of the seamed graphs Δ and Δ'. An isomorphism of the seamed surface onto itself is called an *automorphism*.

2.2. **Covering of a surface by seamed surfaces.** *A covering of degree* n of a surface Ω (generally speaking with boundary) by a seamed surface $\tilde{\Omega} = (\tilde{S}, \tilde{\Delta}, \tilde{\varphi})$ is a continuous map $f : \tilde{\Omega}_{\tilde{\varphi}} \to \Omega$ such that:

1) $f(\tilde{\Omega}_{\tilde{\varphi}}) = \Omega$;

2) f maps $\tilde{\Delta}$ onto $\partial\Omega$ and the restriction of f to every edge is a local homeomorphism;

3) there is only a finite set of points $p \in \Omega \setminus \partial\Omega$ such that the number of their pre-images is strictly smaller n. These points are called *internal critical values.* All other pointe of Ω have exactly n pre-images.

4) if Σ is the set of all vertexes of $\tilde{\Delta}$, then $f^{-1}(f(\Sigma)) = \Sigma$.

The preimage $f^{-1}(u)$ of a small loop u around a point $p \in \Omega \setminus \partial\Omega$ decomposes into several simple contours $\tilde{u}_1, ..., \tilde{u}_m$. The *topological type* of f at p is the non-ordered set of numbers $\alpha = (n_1, ..., n_m)$, where n_i's is the degree of the restriction of the map f onto $\tilde{u}_i$. Clearly, the sum of all n_i is equal to n. Hence, one may treat α as a Young diagram of degree n. The group of automorphisms $\mathsf{Aut}(\alpha)$ consists of auto-homeomorphisms of the set $\tilde{u}_1 \cup ... \cup \tilde{u}_m$ which commute with f.

From now on, we assume that $\partial\Omega$ *is oriented.* Next we define the topological type of any boundary point $q \in \partial\Omega$. Consider a small embedded segment $l \subset \Omega$ surrounding q, i.e. a segment such that $l \cap \partial\Omega = \partial l$ and l cuts out from Ω a disk containing q. The orientation of $\partial\Omega$ determines an order on the segment end-points v_1 and v_2. The segment pre-image $f^{-1}(l)$ forms *a bipartite graph of degree* n, i.e. a graph with vertices divided into two sets $V_1 = f^{-1}(v_1)$ and $V_2 = f^{-1}(v_2)$, and with n edges E connecting vertices from V_1 to vertices from V_2.

The topological equivalence class of the bipartite graph (V_1, E, V_2) is called the *topological type of* q. The point q is *critical* if at least one of the connected components of the graph (V_1, E, V_2) has more than 2 vertices.

An isomorphism of coverings $f : \tilde{\Omega}_{\tilde{\varphi}} \to \Omega$ and $f' : \tilde{\Omega}'_{\tilde{\varphi}'} \to \Omega$ is an isomorphism of the seamed surfaces $F : \Omega_{\varphi} \to \Omega'_{\varphi'}$ such that $f'F = f$. If $f = f'$, then the isomorphism is called an *automorphism.* Automorphisms of the covering f form a group $\mathsf{Aut}(f)$. The groups of automorphisms of isomorphic coverings are isomorphic.

2.3. **Hurwitz numbers.** Denote by $\mathcal{A}_n$ the set of Young diagrams Δ of degree $|\Delta| = n$. Denote by A_n the vector space with basis $\mathcal{A}_n$. Denote by $\mathcal{B}_n$ the set of isomorphism classes of the bipartite graphs with n edges. Denote by B_n the vector space with basis $\mathcal{B}_n$.

Consider a triple $(\Omega, \Omega_a, \Omega_b)$ consisting of a surface Ω, of a finite set of its internal points Ω_a and of a finite set of its boundary points Ω_b. Put $V_\Omega = (\bigotimes_{p\in\Omega_a} A_p) \bigotimes (\bigotimes_{q\in\Omega_b} B_q)$, where A_p and B_q are copies of A_n and B_n.

Consider maps $\alpha : \Omega_a \to \mathcal{A}_n$, $\beta : \Omega_b \to \mathcal{B}_n$ and put $\alpha_p = \alpha(p)$, $\beta_q = \beta(q)$. Denote by $Cov(\Omega, \alpha, \beta)$ the set of isomorphism classes of coverings of the surface Ω by the seamed surfaces $(\tilde{\Omega}, \tilde{\Delta}, \tilde{\varphi})$, with critical values contained in $\Omega_a \cup \Omega_b$ and topological types $\alpha_p \in \mathcal{A}_n$, $\beta_q \in \mathcal{B}_n$ for all $p \in \Omega_a$, $q \in \Omega_b$.

Following [2] we call the number

$$H(\Omega, \alpha, \beta) = \sum_{[f]\in Cov(\Omega,\alpha,\beta)} 1/|\mathsf{Aut}([f])|; \quad |\mathsf{Aut}([f])| = |\mathsf{Aut}(f)| \quad for \quad f \in [f].$$

Hurwitz number of degree n.

For $\partial\Omega = \emptyset$ these numbers coincide with the standard Hurwitz numbers [9], [7]. The correspondance $(\Omega, \alpha, \beta) \mapsto H(\Omega, \alpha, \beta)$ gives rise to a family of linear functionals $\mathcal{H} = \{\Phi_\Omega : V_\Omega \to \mathbb{R}\}$, which (in accordance with [3],[4]) do not depend on the orientation of $\partial\Omega$ and form a Klein topological field theory in the seence of [2].

2.4. **Extended and asymptotic Hurwitz numbers.** We call the Young diagram $\tilde{\Delta}$ of degree n obtained from a Young diagram Δ of degree $m \leq n$ by adding $n-m$ rows of length one a *standard extension of degree n* of Δ. Put

$$\rho_n^A(\Delta) = \frac{|\operatorname{Aut}(\tilde{\Delta})|}{|\operatorname{Aut}(\Delta)||\operatorname{Aut}(\tilde{\Delta}\setminus\Delta)|}\tilde{\Delta}.$$

The correspondence $\Delta \mapsto \rho_n^A(\Delta)$ gives rise to a homomorphism of vector spaces $\rho_n^A : A_m \to A_n$.

We call a graph *simple*, if every connected components has precisely two vertices. By a *the standard extension* of a graph Γ we mean any graph obtained from Γ by adding simple connected components and/or by adding to Γ copies of some edges.

Denote by $\mathcal{E}_n(\Gamma)$ the set of all standard extensions of degree n of the graph Γ. Put by

$$\rho_n^B(\Gamma) = \sum_{\tilde{\Gamma}\in\mathcal{E}_n(\Gamma)} \frac{|\operatorname{Aut}(\tilde{\Gamma})|}{|\operatorname{Aut}(\Gamma)||\operatorname{Aut}(\tilde{\Gamma}\setminus\Gamma)|}\tilde{\Gamma}$$

at $n \geq |\Gamma|$ and $\sigma_n(\Gamma) = 0$ at $n < |\Gamma|$, see [20]. The correspondence $\Gamma \mapsto \rho_n^B(\Gamma)$ gives rise to a homomorphism of vector spaces $\rho_n^B : B_m \to B_n$.

A *free Hurwitz number of degree n* is the number

$$H_n^{fr}(\Omega, \{\alpha_p\}, \{\beta_q\}) = H(\Omega, \{\rho_n^A(\alpha_p)\}, \{\rho_n^B(\beta_q)\}).$$

Here we agree that $H(\Omega, \{\rho_n^A(\alpha_p)\}, \{\rho_n^B(\beta_q)\}) = 0$ if degree of some diagram α_p or some graph β_q is bigger than n. The free Hurwitz number $H_n^{fr}(\Omega, \{\alpha_p\}, \{\beta_q\})$, whose degree is equal to the maximum of degrees of the Young diagrams $\{\alpha_p\}$ and bipartite graphes $\{\beta_q\}$, is called *extended Hurwitz number.* (For classical case $\{\beta_q\} = \emptyset$ this construction was suggested in [23]).

The infinite sequence of free Hurwitz numbers

$$H^{as}(\Omega, \{\alpha_p\}, \{\beta_q\}) = (H_1^{fr}(\Omega, \{\alpha_p\}, \{\beta_q\}), H_2^{fr}(\Omega, \{\alpha_p\}, \{\beta_q\}), \dots)$$

is called *asymptotic Hurwitz number.*

3. Algebra of asymptotic Hurwitz numbers

3.1. **Algebra of Hurwitz numbers of the sphere.** Consider a sphere S with two marked points $S_a = \{p_1, p_2\}$. Define a non degenerate symmetric bilinear form $(.,.)_A : A_n \times A_n \to \mathbb{C}$ by the following formula

$$(\alpha_1, \alpha_2)_A = H(S, \{\alpha_1, \alpha_2\}) = \frac{\delta_{\alpha_1,\alpha_2}}{|\operatorname{Aut}(\alpha_1)|}, \text{ for } \alpha_1, \alpha_2 \in \mathcal{A}_n.$$

Consider a sphere S with three marked points $S_a = \{p_1, p_2, p_3\}$. Then, the equality

$$(\alpha_1 * \alpha_2, \alpha_3)_A = H(S, \{\alpha_1, \alpha_2, \alpha_2\}), \qquad \alpha_1, \alpha_2, \alpha_3 \in \mathcal{A}_n$$

gives rise to a binary operation $*: A_n \times A_n \to A_n$, which makes into A_n *a commutative Frobenius algebra* [7].

The diagram $\mathfrak{e}_n^A \in \mathcal{A}_n$ with all rows of length one is the identity element of this algebra. Besides, $(\alpha_1, \alpha_2)_A = l_A(\alpha_1 * \alpha_2)$, where $l_A : A_n \to \mathbb{C}$ is a linear functional equal to $\frac{1}{n!}$ on $\mathfrak{e}_n^A$ and vanishing on all other Young diagrams.

In accordance with [7], the Hurwitz number corresponding to any sphere S is with k marked points equal to

$$H(S, \{\alpha_1, \dots, \alpha_k\}) = l_A(\alpha_1 * \dots * \alpha_k), \quad \alpha_1, \dots, \alpha_k \in \mathcal{A}_n.$$

An infinite-dimensional version of the algebra A_n is the Ivanov-Kerov algebra A_∞. This algebra is the center of the semi-group algebra of the semi-group D_∞ ([10]). The semi-group D_∞ consists of pairs (d, σ), where d is a subset of the natural numbers $\mathbb{Z}_{>0}$ and $\sigma : d \to d$ is a permutation. Multiplication is given by the formula $(d_1, \sigma_1)(d_2, \sigma_2) = (d_1 \cup d_2, \sigma_1\sigma_2)$.

The conjugation class of an element (d, σ) of the group S_∞ of finite permutations of natural numbers $\mathbb{Z}_{>0}$ is described by a Young diagram. Correspond to the Young diagram Δ the orbit $[\Delta] = \{(d, \sigma) | [d, \sigma] = \Delta\}$. The sums $\sum\limits_{(d,\sigma)\in[\Delta]} (d, \sigma)$, where $\Delta \in \mathcal{A} = \bigcup\limits_{n=1}^{\infty} \mathcal{A}_n$, generate to the center A_∞ of the semi-group algebra of the semi-group D_∞. Denote by $\circ$ the multiplication in A_∞.

Introduce a multiplication $\Delta_1 *_n \Delta_2 = \rho_n^A(\Delta_1) * \rho_n^A(\Delta_2)$ between two Young diagrams Δ_1, Δ_2 of degree not higher than n. Then, in accordance with [18]

$$\Delta_1 \circ \Delta_2 = \sum_{n=\max\{|\Delta_1|,|\Delta_2|\}}^{\infty} \{\Delta_1\Delta_2\}_n,$$

where $\{\Delta_1\Delta_2\}_n = \Delta_1 *_n \Delta_2$ for $n = \max\{|\Delta_1|, |\Delta_2|\}$ and

$$\{\Delta_1\Delta_2\}_n = \Delta_1 *_n \Delta_2 - \sum_{k=\max\{|\Delta_1|,|\Delta_2|\}}^{n-1} \rho_n^A(\{\Delta_1\Delta_2\}_k)$$

for $n > \max\{|\Delta_1|, |\Delta_2|\}$.

In particulary, $\{\Delta_1\Delta_2\}_n = 0$ for $n > |\Delta_1| + |\Delta_2|$.

Let us define a binary operation on the set of sequences $A^{as} = \{(a^1, a^2, \dots) | a^n \in A_n\}$ by $(a_1^1, a_1^2, \dots) * (a_2^1, a_2^2, \dots) = (a_1^1 * a_2^1, a_1^2 * a_2^2, \dots)$. Associate to the element $a \in A_n$ the sequence of numbers $a^{as} = (a^1, a^2, \dots) \in A^{as}$, where $a^i = 0$ at $i < n$ and $a^i = \rho_i^A(a)$ at $i \geq n$.

Theorem 3.1. *The correspondence $a \mapsto a^{as}$ gives rise to the monomorphism of algebras $\rho_\uparrow^A : A_\infty \to A^{as}$.*

Proof. It suffices to check the statement of the theorem for Young diagrams. In this case, the claim follows from the identity

$$\Delta_1 *_n \Delta_2 = \{\Delta_1\Delta_2\}_n + \sum_{k=\max\{|\Delta_1|,|\Delta_2|\}}^{n-1} \rho_n^A(\{\Delta_1\Delta_2\}_k).$$

□

Corollary 3.1. *The homomorphism $\rho^A_\uparrow : A_\infty \to A^{as}$ generates an isomorphic between A^{as} closure of the algebra A_∞.*

Proof. The homomorphism $\rho^A_\uparrow : A_\infty \to A^{as}$ is continued to the monomorphism $\bar\rho^A_\uparrow : \bar A_\infty \to A^{as}$, where $\bar A_\infty = \{\sum\limits_{i=1}^{\infty} a_i | a_i \in A_i\}$ is a closure of the algebra A_∞. Moreover, $\rho^A_\uparrow : \bar A_\infty \to A^{as}$ is an epimorphism. □

Consider the linear operator $l^{as}_A : A^{as} \to \mathbb{C}^\infty = \{(c_1, c_2, \dots)|c_i \in \mathbb{C}\}$, where $l^{as}_A(a^1, a^2, \dots) = (l_A(a^1), l_A(a^2), \dots)$, and the bilinear operator $(a_1, a_2)_A : A^{as} \times A^{as} \to \mathbb{C}^\infty$, where $(a_1, a_2)_A = l^{as}_A(a_1 * a_2)$. Then, from theorem 3.1 follows

Theorem 3.2. *The multiplication in the algebra $A^{as}_\uparrow$ is determined by the equality*

$$(a_1 * a_2, a_3)_A = H^{as}(S, \{a_1, a_2, a_3\}).$$

Moreover,

$$H^{as}(S, \{a_1, \dots, a_k\}) = l^{as}_A(a_1 * \dots * a_k), \quad a_1, \dots, a_k \in A^{as}_\uparrow.$$

Associate with the element $a \in A_\infty$ the sum $l^\Sigma_A(a) = \sum\limits_{n=0}^{\infty} l_A(a^n)$, where $(a^1, a^2, \dots) = \rho^A_\uparrow(a)$.

Lemma 3.1. *The sum $l^\Sigma_A(\rho_\uparrow(a))$ absolutely converges for any $a \in A_\infty$, and $l^\Sigma_A \rho_\uparrow = el_A$.*

Proof. It suffices to check the statement of the lemma for Young diagrams. In this case, $l_A(\Delta) = 0$ unless $\Delta = \mathfrak{e}^A_n$, when

$$l_A(\mathfrak{e}^A_n) = \sum_{k=n}^{\infty} \frac{k!}{k!n!(k-n)!} = \frac{1}{n!} \sum_{k=n}^{\infty} \frac{1}{(k-n)!} = \frac{e}{n!}.$$

□

3.2. **Algebra of boundary Hurwitz numbers of the disk.** We understand by *boundary Hurwitz numbers* the Hurwitz numbers of coverings of the disk without internal critical values.

Denote by $\star : B_n \to B_n$ the involution induced by changing the orientation of the graphs: $\star(V_1, E, V_2) = (V_1, E, V_2)^\star = (V_2, E, V_1)$.

Consider a disk D with two marked points $D_b = \{q_1, q_2\}$. Then, the equality

$$(\beta_1, \beta_2)_B = H(D, \{\beta_1, \beta_2\}) = \frac{\delta_{\beta_1, \beta_2^\star}}{|\operatorname{Aut}(\beta_1)|}, \qquad \beta_1, \beta_2 \in \mathcal{B}_n$$

gives rise to a non-degenerated symmetric bilinear form $(.,.)_B : B_n \times B_n \to \mathbb{C}$.

Consider a disk D with three marked points $D_b = \{q_1, q_2, q_3\}$. Then, the equality

$$(\beta_1 * \beta_2, \beta_3)_B = H(D, \{\beta_1, \beta_2, \beta_2\}), \qquad \beta_1, \beta_2, \beta_3 \in \mathcal{B}_n$$

gives rise to a binary operation $B_n \times B_n \to B_n$, which makes B_n into generically speaking non-commutative Frobenius algebra [4].

Following [3],[4], we describe the algebra B_n in terms of graphs. It is induced by the set of bipartite graphs $\mathcal{B}_n$ of degree n. Multiplication is defined as follows. Let (V_1, E, V_2)

и (V_1', E', V_2') be a pair of bipartite graphs with n edges. Denote by $Hom(V_2, V_1')$ the set of maps $\chi : V_2 \to V_1'$, which preserve the valency of vertices. Associate with each map the bipartite graph (V_2, E_χ, V_1') with the edges connecting only the vertices v and $\chi(v)$, where $v \in V_2$, the number of edges connecting v and $\chi(v)$ being equal to the valency of the vertex v.

Call the subset $F \subset E \times E'$ compatible with χ, if the restriction of the natural projections $E \times E' \to E$, $E \times E' \to E'$ onto F are in one-to-one correspondence and $\chi(V_2(e)) = V_1(e')$ for any $(e, e') \in F$. Denote by M_χ the set of such F's. Associate with $F \in M_\chi$ the bipartite graph $(V_1, \overline{F}, V_2')$, its edges being pairs of edges $(e, e') \in F$ glued together at the points $V_2(e)$ and $V_1(e')$. Denote by $\mathsf{Aut}_F(V_1, \overline{F}, V_2') \subset \mathsf{Aut}(V_1, \overline{F}, V_2')$ the subgroup consisting of the automorphisms, inducing on the set E an automorphism of the graph (V_1, E, V_2).

Now construct the map $\mathcal{B}_n \times \mathcal{B}_n \to B_n$ by putting

$$[(V_1, E, V_2)] * [(V_1', E', V_2')] = \sum_{\chi \in Hom(V_2, V_1')} \sum_{F \in M_\chi} \frac{|\mathsf{Aut}((V_2, \overline{F}, V_1'))|}{|\mathsf{Aut}_F((V_1, \overline{F}, V_2'))|} [(V_1, \overline{F}, V_2')].$$

Extending it by linearity, we obtain a binary operation that makes of B_n into an algebra.

This operation has a simple geometrical meaning. The product $[(V_1, E, V_2)] * [(V_1', E', V_2')]$ is contributed by the valency-preserving identifications of vertices from V_2 with those from V_1'. As a result of such identification, there emerges "a singular graph" with vertices $V_1 \cup V_2'$ and edges intersecting on "the set of singularities" $V_2 = V_1'$. The product, that is defined, is a linear combination of "resolutions" of these singularities by gluing pairs of edges coming into the common vertex from (V_1, E, V_2) and (V_1', E', V_2').

The sum

$$\mathfrak{e}_n^B = \sum_{E \in \mathcal{E}_n} \frac{E}{|\mathsf{Aut}(E)|}$$

over the set of all simple graphs of degree n is the identity element of the algebra B_n. Besides, $(\beta_1, \beta_2)_B = l_B(\beta_1 * \beta_2)$, where $l_B : B_n \to \mathbb{C}$ is the linear functional equal to $\frac{1}{|\mathsf{Aut}(\Gamma)|}$ on the simple graphs Γ and vanishing on all other graphs.

In accordance with [4] the Hurwitz number corresponding to the disk with marked boundary points is given by the formula

$$H(D, \{\beta_1, \ldots, \beta_k\}) = l_B(\beta_1 * \cdots * \beta_k), \quad \beta_1, \ldots, \beta_k \in \mathcal{B}_n.$$

We use the algebras B_n of the bipartite graphs of fixed degree in order to construct an algebra on the vector space B_∞ generated by all bipartite graphs.

Introduce multiplication of the graphs Γ_1, Γ_2 of degree not greater than n: $\Gamma_1 *_n \Gamma_2 = \rho_n^B(\Gamma_1) * \rho_n^B(\Gamma_2)$.

Define the binary operation on the vector space B_∞, requiring that

$$\Gamma_1 \circ \Gamma_2 = \sum_{n=\max\{|\Gamma_1|, |\Gamma_2|\}}^{\infty} \{\Gamma_1 \Gamma_2\}_n,$$

where $\{\Gamma_1\Gamma_2\}_n = \Gamma_1 *_n \Gamma_2$ for $n = \max\{|\Gamma_1|, |\Gamma_2|\}$ and

$$\{\Gamma_1\Gamma_2\}_n = \Gamma_1 *_n \Gamma_2 - \sum_{k=\max\{|\Gamma_1|,|\Gamma_2|\}}^{n-1} \rho_n^B(\{\Gamma_1\Gamma_2\}_k) \quad \text{for } n > \max\{|\Gamma_1|, |\Gamma_2|\}.$$

Lemma 3.2. *The algebra B_∞ is associative and $\{\Gamma_1\Gamma_2\}_n = 0$ for $n > |\Gamma_1| + |\Gamma_2|$.*

Proof. It follows from the definitions that

$$\Gamma_1 \circ \Gamma_2 \circ \Gamma_3 = \sum_{n=\max\{|\Gamma_1|,|\Gamma_2|,|\Gamma_3|\}}^{\infty} \{\Gamma_1\Gamma_2\Gamma_3\}_n,$$

where $\{\Gamma_1\Gamma_2\Gamma_3\}_n \subset B_n$, $\{\Gamma_1\Gamma_2\Gamma_3\}_n$ is invariant under any permutations of Γ_i. This implies associativity of the algebra B_∞. The equality

$$\Gamma_1 *_n \Gamma_2 = \sum_{k=\max\{|\Gamma_1|,|\Gamma_2|\}}^{n-1} \rho_n^B(\{\Gamma_1\Gamma_2\}_k)$$

for $n > \max\{|\Gamma_1|, |\Gamma_2|\}$ follows from the description of the product $*$ above. □

Define on the set of sequences $B^{as} = \{(b^1, b^2, \dots)|b^n \in B_n\}$, a binary operation

$$(b_1^1, b_1^2, \dots) * (b_2^1, b_2^2, \dots) = (b_1^1 * b_2^1, b_1^2 * b_2^2, \dots).$$

Correspond to an element $b \in B_n$ a sequence of numbers $b^{as} = (b^1, b^2, \dots) \in B^{as}$, where $b^i = 0$ at $i < n$ and $b^i = \rho_i^B(a)$ for $i \geq n$.

Theorem 3.3. *The correspondence $b \mapsto b^{as}$ gives rise to a monomorphism of algebras $\rho_\uparrow^B : B_\infty \to B^{as}$.*

Proof. It suffices to check the claim of the lemma for graphs. In this case, it follows from the equality

$$\Gamma_1 *_n \Gamma_2 = \{\Gamma_1\Gamma_2\}_n + \sum_{k=\max\{|\Gamma_1|,|\Gamma_2|\}}^{n-1} \rho_n^B(\{\Gamma_1\Gamma_2\}_k).$$

□

Corollary 3.2. *The homomorphism $\rho_\uparrow^B : B_\infty \to B^{as}$ generates an isomorphic between B^{as} closure of the algebra B_∞.*

Proof. The homomorphism $\rho_\uparrow^B : B_\infty \to B^{as}$ is continued to the monomorphism $\bar{\rho}_\uparrow^B : \bar{B}_\infty \to B^{as}$, where $\bar{B}_\infty = \{\sum_{i=1}^{\infty} a_i | a_i \in B_i\}$ is a closure of the algebra B_∞. Moreover, $\rho_\uparrow^B : \bar{B}_\infty \to B^{as}$ is an epimorphism. □

Consider the linear functional $l_B^{as} : B^{as} \to \mathbb{C}^\infty$, where $l_B^{as}(b^1, b^2, \dots) = (l_B(b^1), l_B(b^2), \dots)$, and the bilinear operator $(b_1, b_2)_B : B^{as} \times B^{as} \to \mathbb{C}^\infty$, where $(b_1, b_2)_B = l_B^{as}(b_1 * b_2)$. Then, from theorem 3.3 it follows that

Theorem 3.4. *The multiplication in the algebra $B_\uparrow$ is defined by the equality*

$$(b_1 * b_2, b_3)_B = H^{as}(D, \{b_1, b_2, b_3\}).$$

Moreover,

$$H^{as}(D, \{b_1, \dots, b_k\}) = l_B(b_1 * \dots * b_k).$$

3.3. **Cardy-Frobenius algebras.** Following [2], [4], [19], recall the definition of a (finite-dimensional) equipped Cardy-Frobenius algebra.

A *Frobenius pair* is a set (C, l^C), where C is a finite-dimensional associative algebra with unit and a linear functional $l^C : C \to \mathbb{C}$, such that the bilinear form $(c_1, c_2)_C = l^C(c_1c_2)$ is non-degenerate.

Casimir element of a Frobenius pair (C, l^C) is the element $K_C = \sum\limits_{i=1}^{n} F^{ij}e_ie_j \in C$, where $\{e_1, \ldots, e_n\}$ is the basis of C and $\{F^{ij}\}$ is the matrix inverse to the matrix $(e_i, e_j)_C$. For involution $\star : C \to C$ we put $K_C^\star = \sum\limits_{i=1}^{n} F^{ij}e_ie_j^\star \in C$.

For Frobenius pairs (A, l^A), (B, l^B) and the linear operator $\phi : A \to B$ denote by $\phi^* : B \to A$ the linear operator, defined by the condition $(\phi^*(b), a)_A = (b, \phi(a))_B$.

A *Cardy-Frobenius algebra* is the data $((A, l^A), (B, l^B), \phi)$, which consists of
1) a commutative Frobenius pair (A, l^A);
2) an arbitrary Frobenius pair (B, l^B);
3) a homomorphism of algebras $\phi : A \to B$ such that the image $\phi(A)$ belongs to the center of the algebra B and $(\phi^*(b'), \phi^*(b''))_A = \mathsf{tr}\, K_{b'b''}$, where the operator $K_{b'b''} : B \to B$ is defined by $K_{b'b''}(b) = b'bb''$.

An *equipped Cardy-Frobenius algebra* is the data $((A, l^A), (B, l^B), \phi, U, \star)$, which consists of
1) the Cardy-Frobenius algebra $((A, l^A), (B, l^B), \phi)$;
2) involutive anti-automorphisms $\star : A \to A$ and $\star : B \to B$ such that $l^A(x^\star) = l^A(x)$, $l^B(x^\star) = l^B(x)$, $\phi(x^\star) = \phi(x)^\star$;
3) an element $U \in A$ such that $U^2 = K_A^\star$ and $\phi(U) = K_B^\star$.

The commutative Frobenius pairs are in one-to-one correspondence [7] with closed topological field theories in the sense of [6].

The Cardy-Frobenius algebras are in one-to-one correspondence with open-closed topological field theories [2]. Open-closed string theories also generate Cardy-Frobenius algebras [15], [13], [16].

Equipped Cardy-Frobenius algebras are in one-to-one correspondence [2] with Klein topological field theories, that are extensions of topological field theories for non-oriented surfaces [2]. Hurwitz numbers of seamed surfaces generate examples of Klein topological field theories [3], [4], [5].

Every real representations of a finite group induces a semi-simple equipped Cardy-Frobenius algebra [14]. There exists a complete classification of the semi-simple equipped Cardy-Frobenius algebras [2].

The above definitions require inverting matrices. Hence, their extension to the infinite-dimensional case requires additional care [19]. We additionally demand that the algebras can be presented as direct (Cartesian) products of finite-dimensional algebras $A = \prod\limits_{\gamma\in\mathfrak{C}} A_\gamma$, $B = \prod\limits_{\gamma\in\mathfrak{C}} B_\gamma$. Instead of functionals on A and B, we will consider the families of functionals $l^A = \{l^A_\gamma : A_\gamma \to \mathbb{C}\}$, $l^B = \{l^B_\gamma : B_\gamma \to \mathbb{C}\}$ such that:
1)(A_γ, l^A_γ) and (B_γ, l^B_γ) are the Frobenius pairs;

2)$\phi(A_\gamma)\in B_\gamma$ and the restrictions ϕ_γ of the homomorphism ϕ onto A_γ give rise to the Cardy-Frobenius algebras $((A_\gamma,l^A_\gamma),(B_\gamma,l^B_\gamma),\phi_\gamma)$;

3) The involution $\star$ preserves the subalgebras A_γ, B_γ and, along with the projections U_γ of the element $U\in A$ onto A_γ, gives rise to the equipped Cardy-Frobenius algebras $((A_\gamma,l^A_\gamma),(B_\gamma,l^B_\gamma),\phi_\gamma,U_\gamma,\star)$.

3.4. **Full algebra of asymptotic Hurwitz numbers.** As was already noted, the sets $((A_n,l_A)$ and (B_n,l_B) form Frobenius pairs. Besides, in [4] were constructed the homomorphism $\phi_n: A_n\to B_n$ and the element U_n such that the set $((A_n,l_A),(B_n,l_B),\phi_n,U_n,\star)$ forms the equipped Cardy-Frobenius algebra.

On the other hand, $A^{as}=\prod\limits_{\gamma=1}^{\infty}A_n$ and $B^{as}=\prod\limits_{\gamma=1}^{\infty}B_n$. The families $\{\phi_n\}$ and $\{U_n\}$ give rise to the homomorphism $\phi^{as}: A^{as}\to B^{as}$ and the element $U^{as}\in A^{as}$. Thus, the set $((A^{as},l^{as}_A),(B^{as},l^{as}_B),\phi^{as},U^{as},\star)$ also forms an equipped Cardy-Frobenius algebra.

In accordance with theorems 3.1 and 3.3 the algebras A^{as} and B^{as} are isomorphic to the algebraic closures of the algebras A_∞ and B_∞.

Theorem 3.5. *If $a\in A_n$, then $\phi^{as}\rho^A_\uparrow(a)=\rho^B_\uparrow\phi_n(a)$.*

Proof. The theorem is equivalent to the relation $\phi_{n+1}\rho^A_n(a)=\rho^B_n\phi_n(a)$, where $\phi_n(a)$ is defined in accordance with [4], by the relation $H(D,\phi_n(a),b)=H(D,a,b)$ for all $b\in B_n$. On the other hand, it follows from the definition of the Hurwitz numbers that $H(D,\rho^B_n(\phi_n(a)),b')=H(D,\rho^A_n(a),b')$ for all $b'\in B_{n+1}$, if $H(D,\phi_n(a),b)=H(D,a,b)$ for all $b\in B_n$. □

Thus, according to corollary 3.1, 3.2

Corollary 3.3. *The algebraic closure of the structure $((A_\infty,l_A),(B_\infty,l_B),\{\phi_n\},\{U_n\},\star)$ forms an equipped Cardy-Frobenius algebra.*

4. Differential equations for generating functions

4.1. **Cut-and-join operators.** Now we construct a representation of the algebras A_∞ and B_∞ as algebras of differential operators acting on the space of functions of infinitely many variables $\{X_{ij}|i,j=1,\dots,\}$ and express the map ϕ in these terms .

The algebra A_∞ is realized as the algebra of the cut-and-join operators $W(\Delta)$ [17],[18], [1]. Recall this construction. We need differential operators of the form

$$D_{ab}=\sum_{e=1}^{\infty}X_{ae}\frac{\partial}{\partial X_{be}}.$$

Associate to the Young diagram $\Delta=[\mu_1,\mu_2,\dots,\mu_k]$ with lines of rows of lengths $\mu_1\geq\mu_2\geq\dots\geq\mu_k$ numbers $m_j=m_j(\Delta)=|\{i|\mu_i=j\}|$ and $\kappa(\Delta)=(|\mathsf{Aut}(\Delta)|)^{-1}=(\prod\limits_j m_j!j^{m_j})^{-1}$. Associate to the Young diagram Δ the *cut-and-join operator*

$$W(\Delta)=\kappa(\Delta):\prod_j(\operatorname{tr}D^j)^{m_j}:,$$

where D is the infinite-dimensional matrix with elements $D_{ab}=\sum_{e=1}^{\infty}X_{ae}\frac{\partial}{\partial X_{be}}$. The symbol : ... : denotes the normal ordering, where all derivatives are placed to the right of all X_{ab} in the product. Denote by W the algebra induced by the operators $W(\Delta)$. Properties of operators $W(\Delta)$ differ a lot from their finite-dimensional counterparts [11].

Theorem 4.1. [18] *The correspondence* $\Delta \mapsto W(\Delta)$ *induces an isomorphism* $\varphi^A : A_\infty \to W$.

4.2. **Graph-operators.** Associate to the monomial $x = X_{a_1b_1} \dots X_{a_nb_n}$ of degree n the bipartite graph $\Gamma(x)$ with edges $\{E_1, \dots, E_m\}$, where the edges E_i and E_j have common left (accordingly, right) vertex if and only if $a_i = a_j$ (accordingly, $b_i = b_j$). Now associate to the graph Γ the *graph-variable*

$$X_\Gamma = \frac{1}{|\,\mathsf{Aut}(\Gamma)|} \sum X,$$

where the sum goes over all monomials x such that $\Gamma(x) = \Gamma$. Denote through X_n the vector space generated by the graph-variables of degree n.

Associate with the operator $D =: D_{a_1b_1} \dots D_{a_nb_n}$: the bipartite graph $\Gamma(\mathcal{D})$ with edges $\{E_1, \dots, E_m\}$, where the edges E_i and E_j have common left (accordingly, right) vertex if and only if $a_i = a_j$ (accordingly, $b_i = b_j$). Now associate with the graph Γ the operator $V[\Gamma] = \frac{1}{|\,\mathsf{Aut}(\Gamma)|} \sum \mathcal{D}$, where the sum goes over all operators $\mathcal{D}$ such that $\Gamma(D) = \Gamma$. We call such operators *graph-operators.*

Define an action of the graph-operators of degree n on the graph-variables of the same degree. The usual action of the graph-operators on the graph-variables leads to diverging summations. Hence, to define a correct differentiation we need to introduce some regularization. To this end, consider, along with the (full) graph-operator and graph-variable $V[\Gamma]$, $X_{[\Gamma']}$ the restricted graph-operator $V^N[\Gamma]$ and graph-variable $X^N_{[\Gamma']}$ defined similarly to the full ones, but with the infinite set of variables $\{X_{ij} | i, j = 1, \dots, \}$ replaced with the finite one $\{X_{ij} | i, j = 1, \dots, N\}$.

Define the action of the graph-operator $V^N[\Gamma]$ on the graph-variable $X^N_{[\Gamma']}$ as the action of the usual differential operator multiplied by $\frac{(N-|R(\Gamma)|)!}{N!}$. One can easily see that $V^N[\Gamma](X^N_{[\Gamma']})$ is a linear combination of the restricted graph-variables $X^N_{[\Gamma"]}$. Moreover, the coefficients of this linear combination are the same if any $N > |E(\Gamma)|$. Now define $V[\Gamma](X_{[\Gamma']}) = \lim\limits_{N\to\infty} V^N[\Gamma](X^N_{[\Gamma']})$. This operation is naturally continued to $|\Gamma| \neq |\Gamma'|$: $V[\Gamma](X_{[\Gamma']}) = 0$ if $|\Gamma| > |\Gamma'|$ and $V[\Gamma](X_{[\Gamma']}) = V[\rho_{|\Gamma'|}(\Gamma)](X_{[\Gamma']})$ at $|\Gamma| < |\Gamma'|$.

Denote through V the algebra generated by the operators $V(\Gamma)$. Define the operation $\circ$ on V requiring that the operator $V[\Gamma_1] \circ V[\Gamma_2]$ acts on all the graph-variables $X[\Gamma]$ as $V[\Gamma_1](V[\Gamma_2](X[\Gamma]))$.

Theorem 4.2. [19] *The correspondence* $\Gamma \mapsto V(\Gamma)$ *establishes the isomorphism* $\varphi^B : B_\infty \to V$.

The cut-and-join operators act on the space of graph-variables by the usual differentiation. Define an homomorphism of algebras $f : W \to V$ requiring that the operator $f(w)$ acts on all the graph-variables as the operator w (we prove below that such an operator exists).

Theorem 4.3. [19] $f\varphi^A = \varphi^B\phi$.

4.3. **The generating function.** The cut-and-join operators are closely related to special generating functions of Hurwitz numbers [18]. We construct now the generating function of Hurwitz numbers for seamed surfaces, which is related to the graph-operators.

Associate with each Young diagram Δ and each bipartite graph Γ formal variables α_Δ and β_Γ.

Fix at the boundary of the disk D a point q and associate to it a bipartite graph Γ. Fix at the boundary of the disk pairwise distinct points $q_1, \dots, q_m$ and associate to them bipartite graphs $\Gamma_1, \dots, \Gamma_m$, where $|\Gamma_i| \leq |\Gamma|$. Fix pairwise distinct internal points $p_1, \dots, p_n$ in the disk disk and associate with them Young diagrams $\Delta_1, \dots, \Delta_n$, where $|\Delta_i| \leq |\Gamma|$. Denote $< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m \parallel \Gamma >$ the extended Hurwitz number corresponding to this set of data. Put $< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m \parallel \Gamma >= 0$, if the degree of at least one Young diagram or one graph from the set is larger than $|\Gamma|$.

Split up the set of Young diagrams $\Delta_1, \dots, \Delta_n$ into the maximal groups of coinciding diagrams. Let $n_1, \dots, n_k$ $(n_1 + \cdots + n_k = n)$ be the numbers of elements in these groups. Split up the set of graphs $\Gamma_1, \dots, \Gamma_m$ into the maximal groups of coinciding graphs. Let $m_1, \dots, m_l$ $(m_1 + \cdots + m_l = m)$ be the numbers of elements in these groups.

Associate to the set of data $(\Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m \parallel \Gamma)$ the monomial

$$< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m \parallel \Gamma > \frac{\alpha_{\Delta_1} \dots \alpha_{\Delta_n}}{n_1! \dots n_k!} \frac{\beta_{\Gamma_1} \dots \beta_{\Gamma_m}}{m_1! \dots m_l!} X_\Gamma,$$

where X_Γ is the graph-variable.

Denote through $Z(\{\alpha_\Delta\}, \{\beta_\Gamma\} | \{X_\Gamma\})$ the formal sum of all such monomials treated as a function of variables α_Δ, β_Γ and X_Γ.

Similarly fix now at the boundary of the disk D two distinct points q, q' and associate to them bipartite graphs Γ, Γ', where $|\Gamma'| \leq |\Gamma|$. Fix at the boundary of the disk pairwise distinct points $q_1, \dots, q_m$ lying outside the arc connecting the points q, q' and associate to them bipartite graphs $\Gamma_1, \dots, \Gamma_m$, where $|\Gamma_i| \leq |\Gamma|$. Fix pairwise distinct internal disk points $p_1, \dots, p_n$ and associate to them Young diagrams $\Delta_1, \dots, \Delta_n$, where $|\Delta_i| \leq |\Gamma|$. Denote $< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m | \Gamma' \parallel \Gamma >$ the extended Hurwitz number corresponding to the set of data. Put $< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m | \Gamma' \parallel \Gamma >= 0$, if the degree of at least one Young diagram or one graph from the set is larger than $|\Gamma|$.

Split up the set of Young diagrams $\Delta_1, \dots, \Delta_n$ into the maximal groups of coinciding diagrams. Let $n_1, \dots, n_k$ $(n_1 + \cdots + n_k = n)$ be the numbers of elements in these groups. Split up the set of graphs $\Gamma_1, \dots, \Gamma_m, \Gamma'$ into the maximal groups of coinciding graphs. Let $m_1, \dots, m_l$ $(m_1 + \cdots + m_l = m + 1)$ be the numbers of elements in these groups.

Associate to the set of data $(\Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m | \Gamma' \parallel \Gamma)$ the monomial

$$< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m | \Gamma' \parallel \Gamma > \frac{\alpha_{\Delta_1} \dots \alpha_{\Delta_n}}{n_1! \dots n_k!} \frac{\beta_{\Gamma_1} \dots \beta_{\Gamma_m}}{m_1! \dots m_l!} X_\Gamma,$$

where X_Γ is the graph-variable.

Denote through $Z_{\Gamma'}(\{\alpha_\Delta\}, \{\beta_\Gamma\} | \{X_\Gamma\})$ the formal sum of all such monomials treated as a function of all variables of a kind of α_Δ, β_Γ and X_Γ.

Theorem 4.4. *The functions Z and Z_Γ satisfy equations*

$$\frac{\partial Z}{\partial \alpha_\Delta} = W(\Delta) Z,$$

$$\frac{\partial Z_{\Gamma'}}{\partial \beta_{\Gamma'}} = V(\Gamma') Z.$$

Proof. The equality $\frac{\partial Z_{\Gamma'}}{\partial \beta_{\Gamma'}} = V(\Gamma')Z$ is equivalent to the system of relations between the numbers $< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m | \Gamma' \parallel \Gamma >$ and $< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m \parallel \Gamma >$, that is,

$$< \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m | \Gamma' \parallel \Gamma >=$$

$$\sum_{i=1}^{k} < \Delta_1, \dots, \Delta_n | \Gamma_1, \dots, \Gamma_m, \Gamma^i > F^{ij} < \Gamma^i \parallel \rho^B_{|\Gamma|}(\Gamma') * \Gamma >,$$

where $\{\Gamma^i\}$ is the set of all bipartite graphs of degree $|\Gamma|$ and F^{ij} is the matrix inverse to $\{< \Gamma^i \parallel \Gamma^j >\}$. These relations are proved in [4] and mean that the Hurwitz numbers are correlators in open-closed topological field theory. The relation $\frac{\partial Z}{\partial \alpha_\Delta} = W(\Delta)Z$ is proved analogously. □

Associate to each connected bipartite graph γ a formal variable q_γ. Consider the algebra Y generated by all variables q_γ. The correspondence $q_\gamma \leftrightarrow X_\gamma$ allows one to interpret the arbitrary graph-variable X_γ as the monomial $q_{\gamma_1} \dots q_{\gamma_k} \in Y$, where $\gamma_1, \dots, \gamma_k$ are the connected components of the graph Γ. The generating functions $Z(\{\alpha_\Delta\}, \{\beta_\Gamma\} | \{X_\Gamma\})$ and $Z_{\Gamma'}(\{\alpha_\Delta\}, \{\beta_\Gamma\} | \{X_\Gamma\})$ then becomes generating functions $\mathcal{Z}(\{\alpha_\Delta\}, \{\beta_\Gamma\} | \{q_\gamma\})$ and $\mathcal{Z}_{\Gamma'}(\{\alpha_\Delta\}, \{\beta_\Gamma\} | \{q_\gamma\})$. The differential operators $W(\Delta)$ and $V(\Gamma)$ acting on the space of graph-variables, after the change of variables become differential operators $\mathtt{W}(\Delta)$, $\mathtt{V}(\Gamma)$, which act on the algebra Y of variables $\{q_\gamma\}$. Theorem 4.4 then becomes

Theorem 4.5. *The functions $\mathcal{Z}$ and $\mathcal{Z}_\Gamma$ satisfy equations*

$$\frac{\partial \mathcal{Z}}{\partial \alpha_\Delta} = \mathtt{W}(\Delta)\mathcal{Z},$$

$$\frac{\partial \mathcal{Z}_{\Gamma'}}{\partial \beta_{\Gamma'}} = \mathtt{V}(\Gamma')\mathcal{Z}.$$

In simplest cases related to coverings by the Klein surfaces [21] this claim is proved in [22] by independent direct combinatorial calculations.

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