# Framed 4-valent Graph Minor Theory I: Intoduction. A Planarity Criterion and Linkless Embeddability

Vassily Olegovich Manturov*

**Abstract**

The present paper is the first one in the sequence of papers about a simple class of *framed* 4-*graphs*; the goal of the present paper is to collect some well-known results on planarity and to reformulate them in the language of *minors*.

The goal of the whole sequence is to prove analogues of the Robertson-Seymour-Thomas theorems for framed 4-graphs: namely, we shall prove that many minor-closed properties are classified by finitely many excluded graphs.

From many points of view, framed 4-graphs are easier to consider than general graphs; on the other hand, framed 4-graphs are closely related to many problems in graph theory.

**Keywords:** graph, 4-valent, minor, planarity, embedding, immersion, Wagner conjecture.

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Some years ago, a milestone in graph theory was established: as a result of series of papers by Robertson, Seymour (and later joined by Thomas) [9] proved the celebrated Wagner conjecture [11]. The Wagner conjecture states that if a class of graphs (considered up to homeomorphism) is minor-closed (i.e., it is closed under edge deletion, edge contraction and isolated node deletion), then it can be characterized by a finite number of excluded minors. For a beautiful review of the subject we refer the reader to L.Lovász [3].

This conjecture was motivated by various pieces of evidence for concrete natural minor-closed properties of graphs, such as knotless or linkless embeddability in $\mathbb{R}^3$, planarity or embeddability in a standardly embedded $S_g \subset \mathbb{R}^3$.

*Bauman Moscow State Technical Unversity 2nd Baumanskaya St.5/1, Moscow, 105005, Russia, and Laboratory of Quantum Topology, Chelyabinsk State University, Brat'ev Kashirinykh street 129, Chelyabinsk 454001, Russia.
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vomanturov@yandex.ru

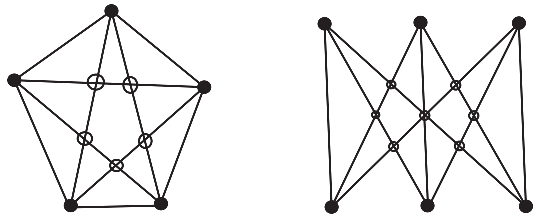

Figure 1: The two Kuratowski graphs, $K_5$ and $K_{3,3}$

Here we say that a property $P$ is *minor-closed* if for every graph $X$ possessing this property every minor $Y$ of $X$ possesses $P$ as well. Later, we shall define the notion of *minor* in a way suitable for framed 4-graphs.

The most famous evidence of this conjecture is the *Pontrjagin-Kuratowski* planarity criterion which states (in a slightly different formulation) that a graph is not planar if and only if it contains one of the two graphs shown in Fig. 1 as a minor.

**Remark 1.** *Throughout the paper (and all subsequent papers in the series), all graphs are assumed to be finite; loops and multiple edges are allowed.*

Among all graphs, there is an important class of four-valent *framed* graphs (or *framed* regular 4-graphs). Here by *framing* we mean a way of indicating which half-edges are opposite at every vertex. Whenever drawing a framed four-valent graph on the plain, we shall indicate its vertices by solid dots, (self)intersection points of edges will be encircled, and the framing is assumed to be induced from the plane: those half-edges which are drawn opposite in $\mathbb{R}^2$ are thought to be opposite. Half-edges of a framed four-valent graph incident to the same vertex which are not *opposite*, are called *adjacent*.

This class of graph is interesting because of its close connection to classical and virtual knot theory [5, 6], homotopy classes of curves on surfaces, see also [1, 2]; for more about virtual knot theory see [7].

From time to time we shall admit some broader class of objects than just framed four-valent graphs. By a 4-*graph* we mean a finite 1-complex with every component either being homeomorphic to a circle or being a graph with all vertices having valency 4; components of a 4-graph homeomorphic to circles will be called *circular components* or *circular edges*; by a *vertex* of a framed 4-graph we mean a vertex of its non-circular component. By a (*non-circular*) edge of a 4-graph we mean an edge of its non-circular component. A 4-graph is *framed* if all non-circular components of it are framed and all circular components of it are oriented.

There are some natural ways to extend the notion of *minor-closed property* to four-valent framed graphs.

**Definition 1.** *A framed* 4*-valent graph* $G'$ *is a* minor *of a framed* 4*-valent graph* $G$ *if* $G'$ *can be obtained from* $G$ *by a sequence of* smoothing *operations* ($\times \to \asymp$ *and* $\times \to )($) *and deletions of connected components.*

**Remark 2.** *Whenever talking about embedding or immersion of a framed* 4*-graph into any* 2*-surface we always assume its framing to be preserved: opposite edges at every crossing should be locally opposite on the surface.*

**Definition 2.** *We say that a framed* 4*-graph* $\Gamma$ *admits a source-sink structure if there is an orientation of all edges of* $\Gamma$ *such that at every vertex of* $\Gamma$ *some two opposite edges are incoming, and the other two are emanating. Certainly, for every connected framed four-valent graph, if a source-sink structure exists, then there are exactly two such structures.*

Moreover, it can be easily seen that if $\Gamma$ admits a source-sink structure then every minor $\Gamma'$ of $\Gamma$ admits a source-sink structure as well. Indeed, the smoothing operation can be arranged to preserve the source-sink structure.

So, it is natural to ask many questions about graphs admitting a source-sink structure.

**Remark 3.** *In the present paper, we restrict ourselves to framed* 4*-graphs admitting source-sink structures. Framed* 4*-graphs not admitting source-sink structures will be considered in subsequent papers.*

Denote by $\Delta$ the following framed 4-graph with 3 vertices: it has 3 vertices $P, Q, R$, and 6 edges $a, a', b, b', c, c'$ such that at vertex $P$ the edges $a$ and $a'$ are opposite and both connect $P$ to $Q$ (in $Q$ they are opposite, as well); $b, b'$ constitute the other pair of opposite edges at $P$; they connect $P$ to $R$, and they are opposite at $R$ as well; finally, $c$ and $c'$ are edges connecting $Q$ and $R$; these edges are opposite both at $Q$ and at $R$.

When drawn immersed in $\mathbb{R}^2$, the graph $\Delta$ contains three pairwise intersecting cycles $(a, a'), (b, b'), (c, c')$; each two of these three cycles intersect transversely at one point; thus, an immersion requires at lease one intersection point for each pair of these two cycles. In Fig. 2 these three immersion points are encircled.

**Definition 3.** *For a framed* 4*-graph* $P$ *by a* loop *we mean either a circular component (also treated as a map* $S^1 \to P$*) or a map* $f : S^1 \to P$ *which is a bijection everywhere except preimages of crossings of* $P$*.*

*A* loop *is a circuit if its image is the whole graph* $P$ *(certainly, only connected framed* 4*-graphs admit circuits).*

*A* loop *(resp., circuit) is* rotating *if at every crossing* $X$ *which has two preimages* $Y_1$ *and* $Y_2$*, the neighbourhoods of* $Y_1$ *is mapped to two* non-opposite *edges.*

*By abuse of notation, we shall say that a loop (a circuit) passes through edges if its image contains these edges.*

**Definition 4.** *Let* $L_1, L_2$ *be two loops of a framed* 4*-graph* $P$*; let* $X$ *be a crossing of* $P$*; we say that* $L_1$ *and* $L_2$ *intersect transversely at* $X$ *if* $L_1$ *passes through a pair of opposite edges at* $X$ *and the same fact hold for* $L_2$*.*

**Definition 5.** *By a* chord diagram *we mean either an oriented circle (*empty*) chord diagram or a cubic graph* $D$ *consisting of an oriented cycle (the* core*) passing through all vertices of* $D$ *such that the complement to it is a disjoint union of edges (*chords*) of the diagram.*

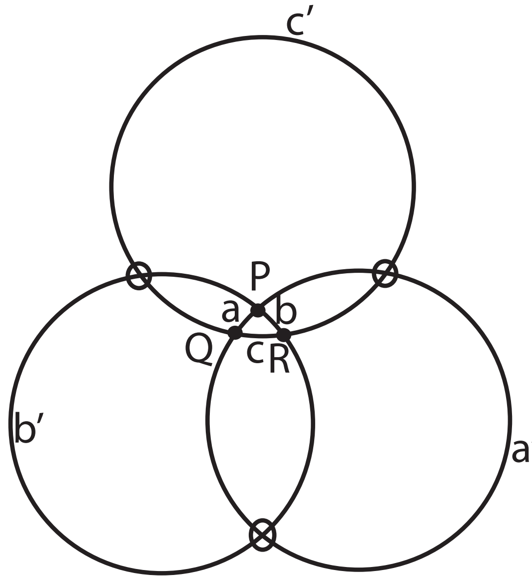


Figure 2: The Graph $\Delta$

An easy exercise (see, e.g. [4]) shows that *every connected framed* 4*-graph admits a rotating circuit.*

Having a circuit $C$ of a framed connected 4-graph $G$, we define the chord diagram $D_C(G)$, as follows. If $G$ is a circle, then $D_C(G)$ is empty. Think of $C$ as a map $f : S^1 \to D$; then we mark by points on $S^1$ preimages of vertices of $G$. Thinking of $S^1$ as a core circle and connecting the preimages by chords, we get the desired cubic graph.

**Remark 4.** *Chord diagrams are considered up to combinatorial equivalence.*

**Remark 5.** *One can associate chord diagrams not only to* rotating circuits*, but in the present paper we restrict ourselves only to rotating circuits and framed* 4*-graphs admitting a source-sink structure.*

The opposite operation (of restoring a framed 4-graph with a source-sink structure from a chord diagram) is obtained by removing chords from the chord diagram and approaching two endpoints of each chord towards each other as shown in Fig. 3.

**Definition 6.** *A chord diagram $D'$ is called a* subdiagram *of a chord diagram $D$ if $D$ can be obtained from $D$ by deleting some chords and their endpoints.*

It follows from the definition that the removal of a chord from a chord diagram results in a smoothing of a framed 4-graph. Consequently, if $D'$ is a subdiagram of $D$, then the resulting framed 4-graph $G(D')$ is a *minor* of $G(D)$.

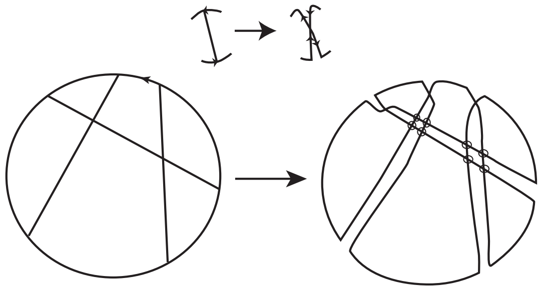

Figure 3: Restoring a framed 4-graph from a chord diagram

Every embedding $i : P \to \mathbb{R}^3$ gives rise to an embedding of every rotating circuit $C$ of $P$: at each vertex where $C$ touches itself we perform a smoothing.

We say that two rotating circuits $C_1, C_2$ sharing no edges are *not transverse* if at every vertex which belongs to both $C_1$ and to $C_2$ the edges incident to $C_1$ are not opposite at this vertex.

Any embedding of a framed 4-valent graph in $\mathbb{R}^3$ is assumed to be smooth in the following sense: in the neighbourhood of every vertex $X$ we require that tangent vectors of opposite half-edges are opposite. Thus, having a framed 4-graph $P$ and an embedding $i : P \to \mathbb{R}^3$, we may assume without loss of generality that the small neighbourhood of every vertex $X$ of $P$ is mapped to a piece of a 2-surface containing $X$. Thus, having two rotating loops $L_1, L_2$ of $P$ with no transverse intersections we can define the associate the disjoint embedding of $L_1$ and $L_2$ in $\mathbb{R}^3$ obtained by local smoothing at some vertices. By abusing notation, we shall talk about *images of loops or circuits* in $\mathbb{R}^3$ meaning the cooresponding smoothings (which represent collection of disjoint curves in $\mathbb{R}^3$.

**Definition 7.** *An embedding $i$ of a framed 4-graph $P$ in $\mathbb{R}^3$ with a source-sink structure is called* linkless *if for every two rotating loops $L_1, L_2$ without transverse intersection the linking number of their images is* 0.

*Analogously, an embedding $i$ of a framed 4-graph $P$ in $\mathbb{R}^3$ with a source-sink structure is* knotless *if the image of the every rotating loop $L$ is unknotted.*

In order to get a closed loop (several closed loops) in $\mathbb{R}^3$, we perform smoothings at some vertices. This means that in the neighbourhood of a vertex where we can perform a smoothing of $X$ and an embedding $i$ gives rise to embeddings of all minors of $P$ defined up to homotopy.

Now we list some *minor* properties of framed 4-valent graphs (the proof is left for the reader):

1. Planarity.

2. Existence of an immersion into a fixed surface $\Sigma$ with no more than $s$ transverse simple intersection points ($s$ fixed).

3. Linkless embeddability (in $\mathbb{R}^3$).

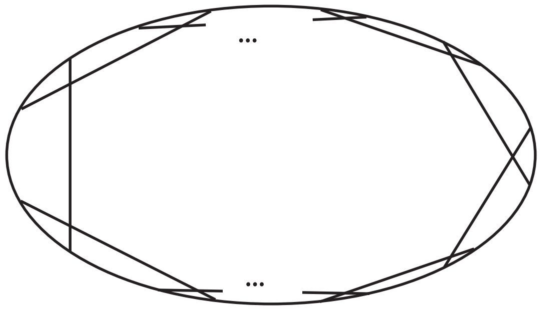

Figure 4: A $(2n+1)$-gon

4. Knotless embeddability (in $\mathbb{R}^3$).

The Main Theorem of the present paper is the following

**Theorem 1.** *Let* $\Gamma$ *be a framed* 4*-graph admitting a source-sink structure. Then the following five conditions are equivalent:*

1. *Every generic immersion of* $\Gamma$ *in* $\mathbb{R}^2$ *requires at least* 3 *additional crossings;*
2. *For every embedding of* $\Gamma$*, there exists a pair of rotating loops with odd linking number.*
3. $\Gamma$ *has no linkless embedding in* $\mathbb{R}^3$*;*
4. $\Gamma$ *is not planar;*
5. $\Gamma$ *contains* $\Delta$ *as a minor.*

*Proof.* Certainly, 3) yields 4): a planar graph has a planar *embedding* which is an immersion with *no additional points*; moreover, a planar embedding is always linkless.

Our next goal is to prove 4) yields 5): the non-planarity of a framed 4-graph with a source-sink structure yields the existence of $\Delta$ as a minor. After that, we see that 5) yields 1): every immersion of $\Delta$ requires at least 3 points, which is obvious, and prove that there for every embedding of $\Delta$ in $\mathbb{R}^3$, there exists a pair of rotating loops without crossing points having *odd linking number* (thus we will get 5)$\longrightarrow$ 2)). The latter automatically means that the embedding is not linkless.

We follow the proof of Vassiliev's conjecture [10] from [4]. Take a rotating circuit $C$ for $\Gamma$; by assumption, $\Gamma$ admits a source-sink structure, thus, the chord diagram $D_C(\Gamma)$ contains a $(2n+1)$-gon $\Delta_{2n+1}$ as a subdiagram, see Fig. 4.

Consequently, the initial graph will have a minor which corresponds to the chord diagram $\Delta_{2n+1}$; we denote this framed 4-graph by $Z_{2n+1}$.

Now, we apply the following fact whose proof is left to the reader as an exercise: $\Delta$ is a minor of $Z_{2n+1}$ for every natural $n$.

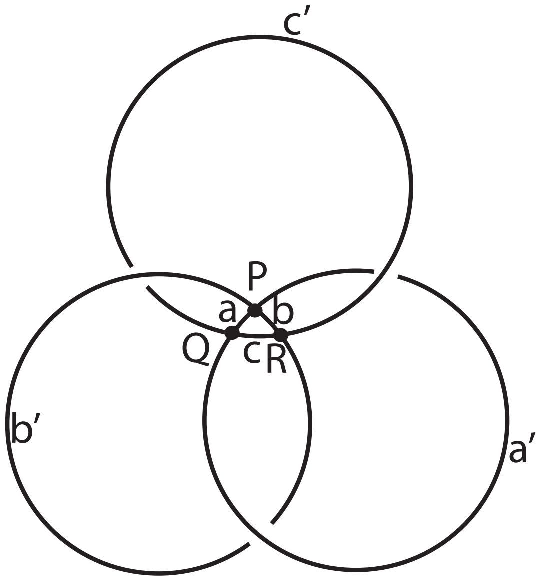


Figure 5: An immersion of $\Delta$ in $\mathbb{R}^3$

Thus, $\Delta$ is a minor of $\Gamma$, which means that 4) yields 5).

Let us now prove that 5) yields 2): there is no linkless embedding of $\Delta$ in $\mathbb{R}^3$; consequently, none exits for $\Gamma$.

Indeed, let us consider the immersion given in Fig. 5.

Let us consider the following four pairs of cycles $F_1 = (a, b, c), F_2 = (a', b', c'), G_1 = (a, b, c'), G_2 = (a', b', c), H_1 = (a, b', c), H_2 = (a', b, c'), I_1 = (a', b, c), I_2 = (a, b', c')$.

For the immersion given in Fig. 5 we see that the linking numbers are $lk(F_1, F_2) = 0$, whence all linking numbers $lk(G_1, G_2), lk(H_1, H_2), lk(I_1, I_2)$ are congruent to 1 modulo 2.

Thus, the sum of these four linking numbers is odd.

Now, linking numbers do not change under isotopy; thus, this sum remains odd when applying isotopy to the immersion given in Fig. 5.

Besides isotopy, we can apply some crossing switches in 3-space. The whole graph $\Delta$ consists of 6 edges; if we apply a crossing switch to an edge with itself (say, $a$ with $a$), it will make no effect in any of the four summands. Now, if we apply a crossing switch for an edge with a dash and a corresponding edge without a prime (say, $a$ and $a'$), this will result in changes modulo 2 for all four summands; thus, the total sum will remain odd.

In the case when we have two letters either both without primes or both with primes (without loss of generality we may assume they are $a$ and $b$), two of four summands will remain the same and the other two will change. Consequently, the parity will remain the same.

Finally, if we apply a crossing switch to some edges which are not opposite at some vertex (without loss of generality, we may assume we deal with $a$ and $b'$), this will change two of four summands: namely, $lk(F_1, F_2)$ and $lk(G_1, G_2)$ will change by one.

Thus, the total parity of the sum of linking numbers will not change.

Thus, we conclude that at least one of these four linking numbers will be odd, which proves that 5) yields 2); moreover, 2) obviously yields 3).

This completes the proof.

$\square$

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