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Two-Sided Bounds for the Complexity of Hyperbolic Three-Manifolds with Geodesic Boundary

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Abstract—We construct an infinite family of hyperbolic three-manifolds with geodesic boundary that generalize the Thurston and Paoluzzi–Zimmermann manifolds. For the manifolds of this family, we present two-sided bounds for their complexity.

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1. INTRODUCTION

Let M be a connected compact three-manifold. A compact two-dimensional polyhedron $P \subset M$ is called a *spine* of M if either $\partial M \neq \varnothing$ and $M \setminus P$ is homeomorphic to $\partial M \times (0,1]$ or $\partial M = \varnothing$ and $M \setminus P$ is homeomorphic to an open ball. The study of the spines of manifolds leads to the notion of *complexity* $c(M)$ of a manifold M [1]. A compact two-dimensional polyhedron P is said to be *almost simple* if the link of each of its points can be embedded in the complete graph K_4 with four vertices. The points whose links are homeomorphic to the graph K_4 are called *true* vertices of the polyhedron P. The spine of a manifold is said to be *almost simple* if it is an almost simple polyhedron. We say that the *complexity* $c(M)$ of a manifold M is equal to n if M has an almost simple spine with n true vertices and has no almost simple spines with a smaller number of true vertices.

Tabulating three-manifolds of a given complexity and obtaining exact values of the complexity for large classes of manifolds provide a natural approach to the problem of their classification. The problem of calculating the complexity of manifolds is quite difficult. To date, the exact values of the complexity are known only for a finite number of tabulated manifolds [2–4] and for several infinite families of manifolds with boundary [5–9] and of closed manifolds [10, 11]. Complexity bounds for some infinite families of manifolds are obtained in [12–15]. A survey of recent results on the complexity of manifolds is given in [16].

In this paper, we consider families of three-manifolds corresponding to pairwise identifications of the faces of a bipyramid. These manifolds include, for example, lens spaces. In Section 2, we present the values of complexity for infinite families of lens spaces. In Section 3, we construct an infinite family of manifolds $M^d_{n,k}$. The manifolds $M^1_{n,k}$ and $M^2_{n,k}$ coincide with the manifolds in [17] and [9], respectively. We prove that $M^d_{n,k}$ are hyperbolic manifolds with geodesic boundary, derive formulas for their volumes, and calculate these volumes for small values of parameters. In Section 4, we establish two-sided bounds for the complexity of the manifolds $M^d_{n,k}$.

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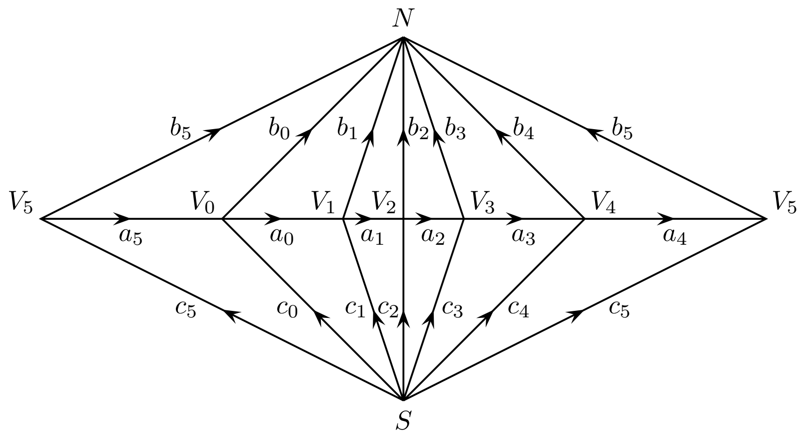

Fig. 1. Bipyramid $\mathcal{B}_6$.

2. LENS SPACES OF KNOWN COMPLEXITY

From several points of view, the simplest class of three-manifolds is formed by manifolds on which one can introduce a spherical geometric structure. These manifolds include lens spaces. Recall their structure. Consider an n-gonal bipyramid $\mathcal{B}_n$, $n \geq 3$, with the notation of vertices and edges as in Fig. 1, which demonstrates the bipyramid $\mathcal{B}_6$. Suppose that the dihedral angles at the edges $b_i = V_iN$ and $c_i = SV_i$ are π, while the dihedral angles at the edges $a_i = V_iV_{i+1}$ are $2\pi/n$, for $i = 0, \dots, n-1$. It is obvious that a bipyramid $\mathcal{B}_n$ with such dihedral angles can be realized in the three-dimensional spherical space $\mathbb{S}^3$, and all its triangular faces are isometric to each other. Let k be an integer such that $1 \leq k \leq n-1$ and $(n, k) = 1$. We define a pairwise identification of the faces of the bipyramid by assuming that an isometry $x^*_{k,i}$ identifies the face $\mathcal{X}_i = a_i b_{i+1} b_i^{-1} = V_iV_{i+1}N$ with the face $\mathcal{X}'_{i+k} = a_{i+k} c^{-1}_{i+k+1} c_{i+k} = V_{i+k}V_{i+k+1}S$ for $i = 0, \dots, n-1$, where all the indices are taken modulo n. The action of the identifications $x^*_{k,i}$ on the faces induces their action on the edges and vertices of the bipyramid. With respect to this action, the edges of the bipyramid are split into $n+1$ equivalence classes:

$$\mathbb{E}_0 = \{b_0, c_k\}, \qquad \dots, \qquad \mathbb{E}_{n-1} = \{b_{n-1}, c_{n-1+k}\}, \qquad \mathbb{E}_n = \{a_0, a_1, \dots, a_{n-1}\},$$

and the vertices are split into two classes: $\{N, S\}$ and $\{V_0, V_1, \dots, V_{n-1}\}$. The quotient space with respect to this equivalence relation is the *lens space* $L_{n,k}$.

All lens spaces of complexity ≤ 13 are described in [4]. Relatively recently, exact values of the complexity have been found for several infinite families of lens spaces.

Theorem 2.1 [10]**.** *The following properties are valid*:

(1) *if* $n \geq 2$, *then* $c(L_{2n,1}) = 2n - 3$;

(2) *if* s *is odd*, t *is even, and* $t > s > 1$, *then* $c(L_{(s+2)(t+1)+1,t+1}) = s + t$;

(3) *if* s *is even*, t *is odd, and* $t > s > 1$, *then* $c(L_{(s+1)(t+2)+1,t+2}) = s + t$.

Theorem 2.2 [11]**.** *If* $n \geq 2$, *then the equality* $c(L_{4n,2n-1}) = n$ *holds.*

3. THE FAMILY OF HYPERBOLIC MANIFOLDS $M^d_{n,k}$

3.1. Construction of manifolds. Let us construct an infinite family of three-manifolds $M^d_{n,k}$ for $n \geq 3$, $0 \leq k \leq n-1$, $1 \leq d \leq n-1$, and $(n, 2-k) = d$. Note that the manifold $M^1_{3,1}$ was presented by Thurston in [18], the manifolds $M^1_{n,k}$ were studied by Paoluzzi and Zimmermann in [17], and the manifolds $M^2_{n,k}$ were studied in [9]. All results of this section for the cases of $d = 1$ and $d = 2$ were established in [17] and [9].

Consider the n-gonal pyramid $\mathcal{B}_n$. We will follow the notation introduced in Section 2 for its vertices, edges, and faces. For every $i = 0, 1, \ldots, n-1$, we define the pairwise identifications $x_{k,i}\colon \mathcal{X}_i \to \mathcal{X}'_{i+k}$ of the faces in accordance with the following traversal order of their boundaries:

$$x_{k,i}\colon\ a_i b_{i+1} b_i^{-1} \to c_{k+i} a_{k+i} c_{k+i+1}^{-1}.$$

The action of the identifications $x_{k,i}$ on the faces induces their action on the edges and vertices of the bipyramid. With respect to this action, the edges of the bipyramid are divided into d equivalence classes:

$$\mathbb{E}_0 = \{a_{dj}, b_{1-k+dj}, c_{k+dj},\ j = 0, 1, \ldots, n'\},$$

$$\mathbb{E}_1 = \{a_{1+dj}, b_{2-k+dj}, c_{1+k+dj},\ j = 0, 1, \ldots, n'\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathbb{E}_{d-1} = \{a_{d-1+dj}, b_{d-k+dj}, c_{d-1+k+dj},\ j = 0, 1, \ldots, n'\},$$

where $n' = n/d - 1$ and all indices are taken modulo n. Denote by $\widetilde{M}^d_{n,k}$ the quotient space obtained by the pairwise identifications $x_{k,i}$, $i = 0, 1, \ldots, n-1$, of the faces of the bipyramid $\mathcal{B}_n$. This space is an orientable pseudomanifold with one singular point, and its Euler characteristic is $\chi(\widetilde{M}^d_{n,k}) = 1 - d + n - 1 = n - d \neq 0$. Cutting out a conical neighborhood of the singular point from $\widetilde{M}^d_{n,k}$, we obtain a compact manifold $M^d_{n,k}$ with a single boundary component.

Let $\mathcal{A}_n$ be a truncated bipyramid obtained from $\mathcal{B}_n$ by truncating (cutting off) all its vertices. The truncation of the vertices N and S leads to two n-gonal faces in $\mathcal{A}_n$, while the truncation of the vertices $V_0, \ldots, V_{n-1}$, to n quadrilateral faces in $\mathcal{A}_n$. After truncating all the vertices, the triangular faces in $\mathcal{B}_n$ turn into hexagonal faces in the truncated bipyramid $\mathcal{A}_n$. The edges of the truncated bipyramid $\mathcal{A}_n$ obtained from the edges a_i, b_i, and c_i of the bipyramid $\mathcal{B}_n$ after truncating its vertices will also be denoted by a_i, b_i, and c_i. The pairwise identifications $x_{k,i}$ of the faces of the bipyramid $\mathcal{B}_n$ naturally induce pairwise identifications $y_{k,i}$ of the hexagonal faces of the truncated bipyramid $\mathcal{A}_n$. Obviously, we can suppose that the quotient space obtained after the pairwise identifications $y_{k,i}$, $i = 0, 1, \ldots, n-1$, coincides with the manifold $M^d_{n,k}$. In this case, the boundary of the manifold is glued from two n-gons and n quadrilaterals.

Proposition 3.1. *The Euler characteristic of the manifold* $M^d_{n,k}$ *is* $\chi(M^d_{n,k}) = d + 1 - n$.

Proof. The required assertion follows from the equalities $\chi(\widetilde{M}^d_{n,k}) = \chi(M^d_{n,k}) + 1 - \chi(\partial M^d_{n,k})$, $\chi(\partial M^d_{n,k}) = 2\chi(M^d_{n,k})$, and $\chi(\widetilde{M}_{n,k}) = n - d$. □

3.2. Hyperbolicity of manifolds.

Theorem 3.1. $M^d_{n,k}$ *is a hyperbolic manifold with geodesic boundary.*

Proof. Let us show that the truncated bipyramid $\mathcal{A}_n$ can be realized in the hyperbolic space $\mathbb{H}^3$ in such a way that the pairwise identifications $y_{k,i}$, $i = 0, 1, \ldots, n-1$, of its faces are realized by isometries of $\mathbb{H}^3$ and for each of the classes $\mathbb{E}_0, \mathbb{E}_1, \ldots, \mathbb{E}_{d-1}$ the sum of dihedral angles over all edges is equal to 2π. This will guarantee the hyperbolicity of the manifold $M^d_{n,k}$.

Let the dihedral angles at the edges b_i and c_i in $\mathcal{A}_n$ be equal to 2α and the dihedral angles at the edges a_i be equal to 2β for all $i = 0, 1, \ldots, n-1$. By construction, every set $\mathbb{E}_i$ consists of n/d edges of type a_i, n/d edges of type b_i, and n/d edges of type c_i. Let us require that the following equality hold:

$$\frac{n}{d}(4\alpha + 2\beta) = 2\pi. \tag{3.1}$$

We also require that two n-gonal faces and n quadrilateral faces in $\mathcal{A}_n$ intersect adjacent faces at right angles. This condition guarantees that the boundary of the hyperbolic manifold is geodesic.

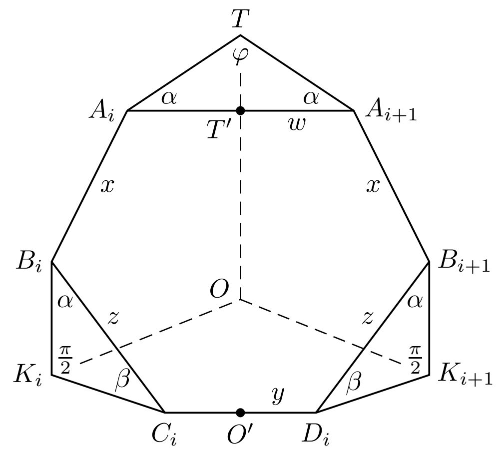

Fig. 2. Polyhedron $\mathcal{C}_n(\alpha,\beta)$.

Let us establish that there exists a realization of $\mathcal{A}_n$ in $\mathbb{H}^3$ that satisfies the above conditions. Let $\mathcal{C}_n(\alpha,\beta)$ be the upper half of the "$1/n$-wedge" of the polyhedron $\mathcal{A}_n$ (Fig. 2). The truncated bipyramid $\mathcal{A}_n$ consists of $2n$ copies of $\mathcal{C}_n(\alpha,\beta)$ (n copies form the upper half of $\mathcal{A}_n$, and another n copies, the lower half). Consider the hexagonal face $A_iB_iC_iD_iB_{i+1}A_{i+1}$ in $\mathcal{C}_n(\alpha,\beta)$ and introduce the following notation for the lengths of its sides:

$$x=|A_iB_i|=|A_{i+1}B_{i+1}|,\qquad y=|C_iD_i|,\qquad z=|B_iC_i|=|D_iB_{i+1}|,\qquad w=|A_iA_{i+1}|.$$

All the angles of the hexagon are equal to $\pi/2$. Let the dihedral angles at the edges A_iB_i and $A_{i+1}B_{i+1}$ equal α and the dihedral angle at the edge C_iD_i equal β. From the formulas for a rectangular hyperbolic hexagon, we obtain

$$\cosh y=\frac{\cosh^2 z+\cosh w}{\sinh^2 z}\qquad\text{and}\qquad \cosh x=\frac{\cosh z\,\cosh w+\cosh z}{\sinh z\,\sinh w}.$$

Set $\varphi=2\pi/n$. By the hyperbolic cosine theorem, from the triangle TA_iA_{i+1} with angles φ, α, and α, we have

$$\cosh w=\frac{\cos^2\alpha+\cos\varphi}{\sin^2\alpha},$$

and from the triangle $K_iB_iC_i$ with angles $\pi/2$, α, and β, we have

$$\cosh z=\cot\alpha\,\cot\beta.$$

Requiring that $\cosh z=\cosh w$ (which automatically implies that $\cosh x=\cosh y$), we find

$$\cot\alpha\,\cot\beta=\frac{\cos^2\alpha+\cos\varphi}{\sin^2\alpha},$$

and so

$$\cot\beta=\cot\alpha+\frac{\cos\varphi}{\sin\alpha\,\cos\alpha}.$$

Let us check that the system of equations

$$\begin{cases} n(2\alpha+\beta)=d\pi,\\ \cot\beta=\cot\alpha+\dfrac{2\cos\varphi}{\sin 2\alpha}\end{cases}\tag{3.2}$$

has solutions for all integer $n \geq 3$. Set $\theta = d\pi/n$. Then $\beta = \theta - 2\alpha$. Hence,

$$\frac{\cos(\theta - 2\alpha)}{\sin(\theta - 2\alpha)} = \frac{\cos\alpha}{\sin\alpha} + \frac{2\cos\varphi}{\sin 2\alpha};$$

thus,

$$\sin 2\alpha\,(\cos\theta\,\cos 2\alpha + \sin\theta\,\sin 2\alpha) = (2\cos^2\alpha + 2\cos\varphi)(\sin\theta\,\cos 2\alpha - \cos\theta\,\sin 2\alpha). \tag{3.3}$$

Recall that in (3.3) we used the notation $\varphi = 2\pi/n$ and $\theta = d\pi/n$. Since $\beta > 0$, it follows that $\alpha < \theta/2$. Let us show that equation (3.3) has a single root α on the interval $(0, \theta/2)$. Introduce the notation $t = \tan\alpha$. Using the formulas $\sin 2\alpha = 2t/(1+t^2)$, $\cos 2\alpha = (1-t^2)/(1+t^2)$, and $\cos^2\alpha = 1/(1+t^2)$, we rewrite (3.3) as

$$\frac{2t}{1+t^2}\left(\frac{1-t^2}{1+t^2}\cos\theta + \frac{2t}{1+t^2}\sin\theta\right) = \left(\frac{2}{1+t^2} + 2\cos\varphi\right)\left(\frac{1-t^2}{1+t^2}\sin\theta - \frac{2t}{1+t^2}\cos\theta\right),$$

which implies

$$t(1-t^2)\cos\theta + 2t^2\sin\theta = (1 + \cos\varphi + t^2\cos\varphi)(\sin\theta - t^2\sin\theta - 2t\cos\theta)$$

and leads to the equation

$$t^4\sin\theta\,\cos\varphi + t^3\cos\theta\,(2\cos\varphi - 1) + 3t^2\sin\theta + t\cos\theta\,(3 + 2\cos\varphi) - \sin\theta\,(1 + \cos\varphi) = 0.$$

Consider the function

$$f_{n,d}(t) = t^4\sin\theta\,\cos\varphi + t^3\cos\theta\,(2\cos\varphi - 1) + 3t^2\sin\theta + t\cos\theta\,(3 + 2\cos\varphi) - \sin\theta\,(1 + \cos\varphi),$$

where $\varphi = 2\pi/n$ and $\theta = d\pi/n$. Since $n \geq 3$ and $d < n$, the following inequality holds:

$$f_{n,d}(0) = -\sin\theta\,(1 + \cos\varphi) = -\sin\frac{d\pi}{n}\left(1 + \cos\frac{2\pi}{n}\right) < 0.$$

Define $\lambda = \tan(\theta/2) = \tan(d\pi/(2n))$. Since $d < n$, we have $\lambda < 1$. Using the equalities $\sin\theta = 2\lambda/(1+\lambda^2)$ and $\cos\theta = (1-\lambda^2)/(1+\lambda^2)$, we obtain

$$\begin{aligned} f_{n,d}(\lambda) &= \frac{2\lambda}{1+\lambda^2}(\lambda^4\cos\varphi + 3\lambda^2 - 1 - \cos\varphi) + \frac{1-\lambda^2}{1+\lambda^2}\big((2\cos\varphi - 1)\lambda^3 + (3 + 2\cos\varphi)\lambda\big) \\ &= \frac{\lambda}{1+\lambda^2}\big[2\lambda^4\cos\varphi + 6\lambda^2 - 2 - 2\cos\varphi + (2\cos\varphi - 1)(\lambda^2 - \lambda^4) + (3 + 2\cos\varphi)(1 - \lambda^2)\big] \\ &= \frac{\lambda}{1+\lambda^2}\big[\lambda^4 + 2\lambda^2 + 1\big] = \frac{\lambda(\lambda^2+1)^2}{1+\lambda^2}. \end{aligned}$$

Since $\lambda > 0$, we have $f_{n,d}(\lambda) > 0$.

Let us show that the function $f_{n,d}(t)$ is strictly increasing on the interval $(0, 1)$. Indeed,

$$\begin{aligned} f'_{n,d}(t) &= 4t^3\cos\varphi\,\sin\theta + 3t^2\cos\theta\,(2\cos\varphi - 1) + 6t\sin\theta + \cos\theta\,(3 + 2\cos\varphi) \\ &= 4t^3\cos\varphi\,\sin\theta + 6t^2\cos\theta\,\cos\varphi - 3t^2\cos\theta + 6t\sin\theta + 3\cos\theta + 2\cos\theta\,\cos\varphi. \end{aligned}$$

If $n = 3$, then $d = 1$, $\varphi = 2\pi/3$, $\theta = \pi/3$, and $f'_{3,1}(t) = -\sqrt{3}t^3 - 3t^2 + 3\sqrt{3}t + 1$. One can easily verify that this polynomial has three real roots and takes positive values on the interval $(0, 1)$. Let

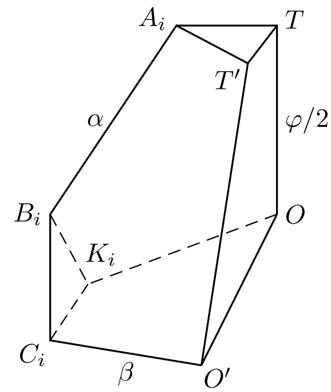

Fig. 3. Polyhedron $\mathcal{D}_n(\alpha,\beta)$.

$n \geq 4$. Then $\varphi = 2\pi/n \in (0,\pi/2]$ and $\theta = d\pi/n \in (0,\pi/2]$, because $d < n$ and d is a divisor of n. Taking into account that $3(1-t^2)\cos\theta > 0$ for $t \in (0,1)$, we obtain $f'_{n,d}(t) > 0$. Hence, equation (3.3) has a single root α on the interval $(0,\theta/2)$.

Since $\beta = \theta - 2\alpha \in (0,\theta)$, the wedge $\mathcal{C}_n(\alpha,\beta)$ is realized as an acute-angled polyhedron in $\mathbb{H}^3$. Moreover, the edge lengths satisfy the equalities $|A_iB_i| = |C_iD_i|$ and $|A_iA_{i+1}| = |B_iC_i| = |D_iB_{i+1}|$ for all i. Hence, the truncated pyramid $\mathcal{A}_n$ can be realized in $\mathbb{H}^3$ in such a way that α and β satisfy (3.1) and all hexagonal faces of the polyhedron $\mathcal{A}_n$ are equal to each other and can be pairwise identified by the isometries $y_{k,i}$. $\square$

The knowledge of the values of α and β allows one to find the volumes of the manifolds $M^d_{n,k}$ in terms of the Lobachevsky function

$$\Lambda(x) = -\int_0^x \ln|2\sin\zeta|\,d\zeta.$$

Proposition 3.2. *The volume of the hyperbolic manifold $M^d_{n,k}$ is given by*

$$\operatorname{vol} M^d_{n,k} = n\biggl[\Lambda\Bigl(\frac{\pi}{n}+\delta_0\Bigr) - \Lambda\Bigl(\frac{\pi}{n}-\delta_0\Bigr) + \Lambda\Bigl(\frac{\pi}{2}+\alpha-\delta_0\Bigr) + \Lambda\Bigl(\frac{\pi}{2}-\alpha-\delta_0\Bigr) + \Lambda\biggl(\frac{d\pi}{n}-2\alpha+\delta_0\biggr) - \Lambda\biggl(\frac{d\pi}{n}-2\alpha-\delta_0\biggr) + 2\Lambda\Bigl(\frac{\pi}{2}-\delta_0\Bigr)\biggr],$$

where

$$\delta_0 = \arctan\frac{\sqrt{\cos^2\alpha - \sin^2\bigl(\frac{\pi}{n}\bigr)\sin^2\bigl(\frac{d\pi}{n}-2\alpha\bigr)}}{\cos\bigl(\frac{\pi}{n}\bigr)\cos\bigl(\frac{d\pi}{n}-2\alpha\bigr)} \in \Bigl[0,\frac{\pi}{2}\Bigr),$$

and $\alpha \in (0, d\pi/(2n))$ is the root of equation (3.3).

Proof. By construction, $\operatorname{vol} M^d_{n,k} = \operatorname{vol}\mathcal{A}_n$. As pointed out above, the polyhedron $\mathcal{A}_n$ consists of $2n$ copies of the wedge $\mathcal{C}_n(\alpha,\beta)$; hence, $\operatorname{vol} M^d_{n,k} = 2n\operatorname{vol}\mathcal{C}_n(\alpha,\beta)$. The plane passing through the points T, T', O', and O splits the wedge $\mathcal{C}_n(\alpha,\beta)$ into two polyhedra that are symmetric to each other with respect to this plane. Denote the polyhedron $O'OTT'A_iB_iC_iK_i$ by $\mathcal{D}_n(\alpha,\beta)$ (Fig. 3). Then $\operatorname{vol} M^d_{n,k} = 4n\operatorname{vol}\mathcal{D}_n(\alpha,\beta)$. To calculate $\operatorname{vol}\mathcal{D}_n(\alpha,\beta)$, it suffices to note that $\mathcal{D}_n(\alpha,\beta)$ is a doubly truncated birectangular tetrahedron with successively adjacent dihedral angles π/n, α, and $d\pi/n - 2\alpha$ and with all other dihedral angles equal to $\pi/2$. The volume of such a tetrahedron is expressed by the formula from [19]; the substitution of our angles into this formula yields the required result (see [9] for more details). $\square$

Volumes of manifolds

d	n	$\mathrm{vol}\,\mathcal{D}_n(\alpha,\beta)$	$\mathrm{vol}\,M^d_{n,k}$	d	n	$\mathrm{vol}\,\mathcal{D}_n(\alpha,\beta)$	$\mathrm{vol}\,M^d_{n,k}$
1	3	0.537665	6.451990	3	6	0.698821	16.771717
1	4	0.714588	11.448760	3	9	0.826332	29.747963
1	5	0.791390	15.827810	3	12	0.866671	41.600235
2	4	0.588490	9.415841	4	8	0.736166	23.557327
2	6	0.784052	18.817257	4	12	0.840879	40.362214
2	8	0.841987	27.007588				

The table presents the volumes of the birectangular truncated tetrahedra $\mathcal{D}_n(\alpha,\beta)$ and the manifolds $M^d_{n,k}$ for small values of n. For the cases of $d = 1$ and $d = 2$, the formulas of volumes and their numerical values are obtained in [20] and [9].

4. COMPLEXITY OF THE MANIFOLDS $M^d_{n,k}$

4.1. Two-sided complexity bounds. In the class of almost simple polyhedra, one distinguishes the subclasses of so-called simple and special polyhedra. A compact polyhedron P is said to be *simple* if the link of each point $x \in P$ is homeomorphic either to a circle (such a point x is said to be *nonsingular*), or to a circle with diameter (such a point x is said to be *triple*), or to the graph K_4 (such a point x, just as in the case of an almost simple polyhedron, is called a *true vertex*). The connected components of the union of all nonsingular points and of the union of all triple points are called the 2-*components* and the *triple lines* of the polyhedron P, respectively. A simple polyhedron is said to be *special* if its triple lines are open 1-cells and its 2-components are open 2-cells. A spine of a manifold is said to be *simple* or *special* if it is a simple or special polyhedron, respectively.

Let M be an irreducible boundary-irreducible three-manifold. A proper annulus in M is said to be *inessential* if either it is parallel to an annulus in ∂M with respect to the boundary or its core circle is contractible in M. The following theorem shows that, in many cases, an almost simple spine of a manifold can be replaced by a special spine without increasing the number of true vertices.

Theorem 4.1 [1]**.** *Suppose that a compact irreducible boundary-irreducible three-manifold M is such that $M \neq D^3, S^3, \mathbb{RP}^3, L_{3,1}$ and all proper annuli in M are inessential. Then, for any almost simple spine P of M, one can find its special spine P_1 with a smaller or the same number of true vertices.*

As a corollary to the theorem, we obtain

Proposition 4.1. *Suppose that a manifold M satisfies the hypotheses of Theorem* 4.1. *Then $c(M) \geq 1 - \chi(M)$.*

Proof. By Theorem 4.1, there exists a special spine P of M with $c(M) \geq 1$ true vertices. Since P is special, the number of its triple lines is equal to $2c(M)$ and each of them is a 1-cell. Let m be the number of its 2-components; then $m \geq 1$. Hence, $\chi(P) = m - c(M)$, which implies that $c(M) \geq 1 - \chi(P)$. If $\partial M \neq \varnothing$, then $\chi(M) = \chi(P)$. If $\partial M = \varnothing$, then $\chi(M) = 0$. In both cases, the required inequality holds. □

Now we present two-sided bounds for the complexity of the manifolds $M^d_{n,k}$.

Theorem 4.2. *The following inequalities hold*:

$$n - d \leq c(M^d_{n,k}) \leq n.$$

Proof. Let us construct a special spine $P^d_{n,k}$ of $M^d_{n,k}$ with n true vertices. We split the bipyramid $\mathcal{B}_n$ into n tetrahedra $\mathcal{T}_i = NSV_iV_{i+1}$, $i = 0, 1, \dots, n-1$ (see Fig. 1). In each $\mathcal{T}_i$, consider the union R_i of the links of all four vertices of the tetrahedron in its first barycentric subdivision. The space $\widetilde{M}^d_{n,k}$ is obtained by gluing together the tetrahedra $\mathcal{T}_0, \ldots, \mathcal{T}_{n-1}$ by affine homeomorphisms of their faces. The gluing of the tetrahedra defines a pseudotriangulation $\mathcal{T}$ of the space $\widetilde{M}^d_{n,k}$ and induces a gluing of the polyhedra R_i, $i = 0, \ldots, n-1$. As a result, we obtain a spine $P^d_{n,k} = \bigcup_i R_i$ of $M^d_{n,k}$, which is special by virtue of [1]. Since every polyhedron R_i is homeomorphic to a cone over the graph K_4, the spine $P^d_{n,k}$ has exactly n true vertices. Hence, $c(M^d_{n,k}) \le n$.

Since the manifold $M^d_{n,k}$ is hyperbolic, it satisfies the hypotheses of Theorem 4.1. By Proposition 3.1, $\chi(M^d_{n,k}) = d + 1 - n$. Thus, by Proposition 4.1, we obtain $c(M) \ge 1 - \chi(M) = n - d$. □

Note that for $d = 1$ and $d = 2$, the exact values of the complexity of the manifolds are known.

Proposition 4.2. *The following equalities hold*:

(1) $c(M^1_{3,1}) = 2$ (*see* [6]);

(2) $c(M^1_{n,k}) = n$ *for* $n \ge 4$ (*see* [8]);

(3) $c(M^2_{n,k}) = n$ *for* $n \ge 6$ (*see* [9]).

4.2. The ε-invariant. A particular case of the Turaev–Viro invariants of three-manifolds is the ε-invariant, which is defined as follows (see [1]). Let P be a special spine of a compact manifold M. Denote by $\mathcal{F}(P)$ the set of all its simple subpolyhedra, including P and the empty set. To every simple polyhedron $Q \subset P$, we assign its ε-weight

$$w_\varepsilon(Q) = (-1)^{V(Q)} \varepsilon^{\chi(Q) - V(Q)},$$

where $V(Q)$ is the number of true vertices of the polyhedron Q, $\chi(Q)$ is its Euler characteristic, and $\varepsilon = (1 + \sqrt{5})/2$ is a solution of the equation $\varepsilon^2 = \varepsilon + 1$. Then the ε-invariant $t(M)$ of M is defined as

$$t(M) = \sum_{Q \in \mathcal{F}(P)} w_\varepsilon(Q).$$

The values of the ε-invariant for the manifolds $M^1_{n,k}$ and $M^2_{n,k}$ are calculated in [8] and [9] by using the special spines $P^1_{n,k}$ and $P^2_{n,k}$.

Proposition 4.3. *The following equalities hold*:

(1) $t(M^1_{n,k}) = (-1)^n \varepsilon^{2-2n} + \varepsilon^{1-n} + 1$;

(2) $t(M^2_{n,k}) = \varepsilon^{3-2n} + \varepsilon^{2-n} + 1$.

As the parameter d increases, the formula for the values of the ε-invariant for the manifolds $M^d_{n,k}$ becomes more complicated, because the number of simple subpolyhedra of the spines $P^d_{n,k}$ increases. The values of the ε-invariant for the manifolds $M^3_{n,k}$ are given in the next theorem.

Theorem 4.3. *The following equality holds*:

$$t(M^3_{n,k}) = (-1)^n \varepsilon^{4-2n} + \varepsilon^{3-n} + 3\varepsilon^{2-\frac{2n}{3}} + 1.$$

Proof. Let us calculate $t(M^3_{n,k})$ by the special spine $P^3_{n,k}$ constructed in Theorem 4.2. To this end, we describe the set of all of its proper simple subpolyhedra. For compactness reasons, if a simple polyhedron $Q \subset P^3_{n,k}$ contains at least one point of a 2-component ξ of the polyhedron $P^3_{n,k}$, then $\xi \subset Q$. Moreover, to describe the subpolyhedron Q, it suffices to point out which 2-components of the polyhedron $P^3_{n,k}$ are contained in Q, because the other points of Q (i.e., the triple points and true vertices of $P^3_{n,k}$ that lie in Q) are uniquely recovered from this information.

By the construction of the special spine $P^3_{n,k}$, its 2-components are in one-to-one correspondence with the edges of the pseudotriangulation $\mathcal{T}$. Since $\mathcal{T}$ contains four edges, the special spine $P^3_{n,k}$ contains four 2-components. Let ξ be the 2-component corresponding to the edge NS of the pseudotriangulation $\mathcal{T}$. Denote by ζ_0, ζ_1, and ζ_2 the 2-components corresponding to the edge classes $\mathbb{E}_0$, $\mathbb{E}_1$, and $\mathbb{E}_2$, respectively. A combinatorial analysis shows that of all possible combinations of 2-components of the polyhedron $P^3_{n,k}$, only four $\{\zeta_0, \zeta_1, \zeta_2\}$, $\{\zeta_0, \zeta_1\}$, $\{\zeta_0, \zeta_2\}$, and $\{\zeta_1, \zeta_2\}$ generate proper simple subpolyhedra, which we denote by Q_ξ, $Q_{\xi\zeta_2}$, $Q_{\xi\zeta_1}$, and $Q_{\xi\zeta_0}$, respectively.

By Proposition 3.1, we have $\chi(P^3_{n,k}) = 4 - n$. Hence, $w_\varepsilon(P^3_{n,k}) = (-1)^n \varepsilon^{4-2n}$. One can easily verify that the polyhedra Q_ξ, $Q_{\xi\zeta_2}$, $Q_{\xi\zeta_1}$, and $Q_{\xi\zeta_0}$ have no true vertices and that $\chi(Q_\xi) = 3 - n$ and $\chi(Q_{\xi\zeta_0}) = \chi(Q_{\xi\zeta_1}) = \chi(Q_{\xi\zeta_2}) = 2 - 2n/3$. By the definition of the ε-invariant, we obtain $t(M^3_{n,k}) = (-1)^n \varepsilon^{4-2n} + \varepsilon^{3-n} + 3\varepsilon^{2-2n/3} + 1$. $\square$

The calculation of ε-invariants has played a significant role in determining the exact values of the complexity of the manifolds $M^1_{n,k}$ and $M^2_{n,k}$ (these values are presented in assertions (2) and (3) of Proposition 4.2). Let us explain this by presenting the main idea of the proof of assertion (2) for the manifolds $M^1_{n,k}$, $n \geq 4$. In this case, the inequality from Theorem 4.2 reads

$$n - 1 \leq c(M^1_{n,k}) \leq n.$$

Arguing by contradiction, we suppose that $c(M^1_{n,k}) = n - 1$. Then, by Theorem 4.1, there exists a special spine P of $M^1_{n,k}$ with $n - 1$ true vertices. Since the Euler characteristic of the manifold $M^1_{n,k}$ coincides with the Euler characteristic of the spine P and is equal to $2 - n$ (Proposition 3.1), the spine P contains exactly one 2-component. Hence, $\mathcal{F}(P) = \{\varnothing, P\}$; calculating $t(M^1_{n,k})$ by the special spine P, we obtain $t(M^1_{n,k}) = (-1)^{n-1}\varepsilon^{3-2n} + 1$. At the same time, according to assertion (1) of Proposition 4.3, we have $t(M^1_{n,k}) = (-1)^n \varepsilon^{2-2n} + \varepsilon^{1-n} + 1$. As shown in [8], the coincidence of the above two expressions for $t(M^1_{n,k})$ is possible only for $n = 3$, which leads to a contradiction.

Similar arguments involving the formula for the ε-invariant from assertion (2) of Proposition 4.3 have made it possible to establish assertion (3) of Proposition 4.2 for the manifolds $M^2_{n,k}$. The main technical difficulty here consists in describing the set of simple subpolyhedra of an arbitrary special spine P of $M^2_{n,k}$ with two 2-components (see [9] for details).

A similar approach to finding the exact value of the complexity for the manifolds $M^3_{n,k}$ faces the problem of describing the set of simple subpolyhedra of an arbitrary special spine of $M^3_{n,k}$ that contains three 2-components.

In view of Theorem 4.2 and Proposition 4.2, we suggest the following conjecture.

Conjecture. *For any $d \geq 3$, there exists an $n(d)$ such that the equality $c(M^d_{n,k}) = n$ holds for all $n \geq n(d)$.*

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