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Trigonometric identities and volumes of the hyperbolic twist knot cone-manifolds

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ABSTRACT

We calculate the volumes of the hyperbolic twist knot cone-manifolds using the Schläfli formula. Even though general ideas for calculating the volumes of cone-manifolds are around, since there is no concrete calculation written, we present here the concrete calculations. We express the length of the singular locus in terms of the distance between the two axes fixed by two generators. In this way the calculation becomes simpler than using the singular locus directly. The volumes of the hyperbolic twist knot cone-manifolds simpler than Stevedore's knot are known. We extend Mednykh's Methods[a]. As an application, we give the volumes of the cyclic coverings over the hyperbolic twist knots.

Keywords: hyperbolic orbifold, hyperbolic cone-manifold, volume, complex distance, twist knot, orbifold covering.

Mathematics Subject Classification 2000: 57M25, 57M27

1. Introduction

Thurston [22, Chapter 5] showed that a holonomy representation h_∞ of the group of a knot K into $\mathrm{PSL}(2,\mathbb{C})$ can be deformed a little bit to give a one-parameter

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[a]Explicit volume formula for cone-manifolds of 3-twist knot in [13]

family $\{h_\alpha\}$ of representations and to give a corresponding one-parameter family $\{C_\alpha\}$ of singular complete hyperbolic manifolds, the hyperbolic *cone-manifolds* of a knot K. Let m be a meridian of K. Kojima [11] showed further that C_α is totally determined by the action of $h_\alpha(m)$ which is a rotation of angle α around the fixed axis of h_α. A point on K of the cone-manifold C_α is in the core of a neighborhood isometric to a cylinder made of an angle α wedge by identifying the two boundaries. The α is called a *cone-angle* along K. A point off K has a neighborhood isometric to a neighborhood in $\mathbb{H}^3$. We consider the complete hyperbolic structure on the knot complement as the cone-manifold structure of cone-angle zero.

As we mentioned, if we increase the cone-angle from zero and if we keep the angle small, we get a one-parameter family of hyperbolic cone-manifolds. Similarly, for a link K having n components, we can get an n-parameter family of hyperbolic cone-manifolds of a link K. In particular, for each two-bridge hyperbolic link, there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ for each link K such that C_α is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$ [18,8,11,19].

Explicit volume formulae for hyperbolic cone-manifolds of knots and links are known only for a few cases. The volume formulae for hyperbolic cone-manifolds of the knot 4_1 [8,11,12,15], the knot 5_2 [13], the link 5_1^2 [16], the link 6_2^2 [17], and the link 6_3^2 [3] have been calculated. In [9] a method of calculating the volumes of two-bridge knot cone-manifolds were introduced but without explicit formulae.

The main purpose of the paper is to find explicit and efficient volume formula for hyperbolic twist knot cone-manifolds. The following theorem gives the formula for T_m for even integers m. For odd integers m, we can replace T_m by T_{-m-1} as explained in Section 2. So, the following theorem actually covers all possible hyperbolic twist knots. But for the volume formula, since the knot T_{2n} has to be hyperbolic, we exclude the case when $n = 0,\ -1$.

Theorem 1.1. *Let T_{2n} be a hyperbolic twist knot. Let $T_{2n}(\alpha)$, $0 \leq \alpha < \alpha_0$ be the hyperbolic cone-manifold with underlying space S^3 and with singular set T_{2n} of cone-angle α. Then the volume of $T_{2n}(\alpha)$ is given by the following formula*

$$\operatorname{Vol}(T_{2n}(\alpha)) = \int_\alpha^\pi \log\left|\frac{A+iV}{A-iV}\right| d\alpha,$$

where for $A = \cot\frac{\alpha}{2}$, V with $\operatorname{Im}(V) \leq 0$ is a zero of the complex distance polynomial $P_{2n} = P_{2n}(V, B)$ which is given recursively by

$$P_{2n} = \begin{cases} \left(\left(4B^4 - 8B^2 + 4\right)V^2 - 4B^4 + 8B^2 - 2\right)P_{2(n-1)} - P_{2(n-2)}, & \text{if } n > 1, \\ \left(\left(4B^4 - 8B^2 + 4\right)V^2 - 4B^4 + 8B^2 - 2\right)P_{2(n+1)} - P_{2(n+2)}, & \text{if } n < -1, \end{cases}$$

with initial conditions

$$\begin{aligned} P_{-2}(V,B) &= \left(2B^2-2\right)V + 2B^2 - 1,\\ P_0(V,B) &= 1,\\ P_2(V,B) &= \left(4B^4-8B^2+4\right)V^2 + \left(2-2B^2\right)V - 4B^4 + 6B^2 - 1, \end{aligned}$$

where $B = \cos\frac{\alpha}{2}$.

2. Twist knots

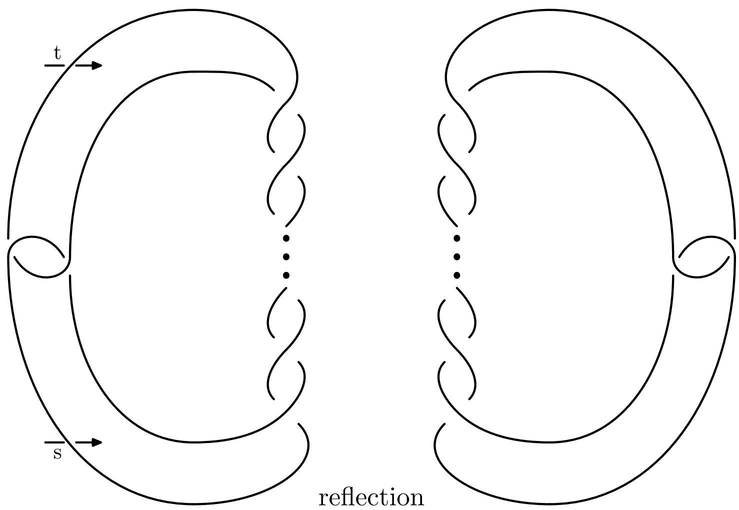

Fig. 1. A twist knot (left) and its mirror image (right)

A knot K is a twist knot if K has a regular two-dimensional projection of the form in Figure 1. For example, Figure 2 is knot 6_1. K has 2 right-handed horizontal crossings and m right-handed vertical crossings. We will denote it by T_m. Note that T_m and its mirror image have the same fundamental group and hence have the same fundamental domain up to isometry in $\mathbb{H}^3$. It follows that $T_m(\alpha)$ and its mirror image have the same fundamental set up to isometry in $\mathbb{H}^3$ and have the same volume. So, we will make no distinction between T_m and its mirror image because we are calculating volumes. Since the mirror image of T_m is equivalent to T_{-m-1}, when m is odd we will think T_{-m-1} as T_m. Hence a twist knot can be represented by T_{2n} for some integer n.

Let us denote by X_m the exterior of T_m in S^3. In [20, Proposition 1], the fundamental group of two-bridge knots is presented. We will use the fundamental group of X_{2n} in [10]. In [10], the fundamental group of X_{2n} is calculated with 2

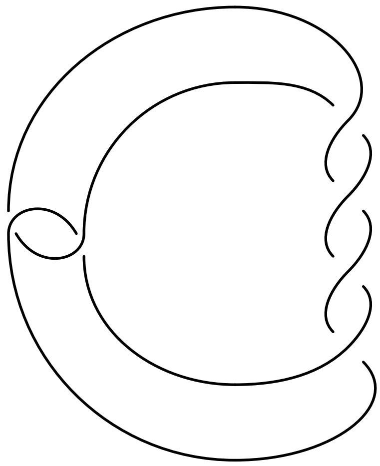

Fig. 2. The knot 6_1

left-handed horizontal crossings as positive crossings instead of two right-handed horizontal crossings. The following proposition is tailored to our purpose.

Proposition 2.1.

$$\pi_1(X_{2n}) = \left\langle s, t \mid swt^{-1}w^{-1} = 1 \right\rangle,$$

where $w = (ts^{-1}t^{-1}s)^n$.

We remark here that s of Proposition 2.1 is the meridian which winds around the bottom arc of the twist knot in Figure 1 and t is the one that does the top arc as in Figure 1.

3. The complex distance polynomial and A-polynomial

Let $R = \mathrm{Hom}(\pi_1(X_{2n}), \mathrm{SL}(2, \mathbb{C}))$. Given a set of generators, s, t, of the fundamental group for $\pi_1(X_{2n})$, we define a set $R\,(\pi_1(X_{2n})) \subset \mathrm{SL}(2, \mathbb{C})^2 \subset \mathbb{C}^8$ to be the set of all points $(\eta(s), \eta(t))$, where η is a representaion of $\pi_1(X_{2n})$ into $\mathrm{SL}(2, \mathbb{C})$. Since the defining relation of $\pi_1(X_{2n})$ gives the defining equation of $R\,(\pi_1(X_{2n}))$ [21], $R\,(\pi_1(X_{2n}))$ is an affine algebraic set in $\mathbb{C}^8$. $R\,(\pi_1(X_{2n}))$ is well-defined up to isomorphisms which arise from changing the set of generators. We say elements in R which differ by conjugations in $\mathrm{SL}(2, \mathbb{C})$ are *equivalent*.

We use two coordinates to give the structure of the affine algebraic set to $R\,(\pi_1(X_{2n}))$. Equivalently, for some $O \in \mathrm{SL}(2, \mathbb{C})$, we consider both η and $\eta' = O^{-1}\eta O$:

For the complex distance polynomial, we use for the coordinates

$$\eta(s)=\begin{bmatrix}(M+1/M)/2 & e^{\frac{\rho}{2}}(M-1/M)/2\\ e^{-\frac{\rho}{2}}(M-1/M)/2 & (M+1/M)/2\end{bmatrix},$$

$$\eta(t)=\begin{bmatrix}(M+1/M)/2 & e^{-\frac{\rho}{2}}(M-1/M)/2\\ e^{\frac{\rho}{2}}(M-1/M)/2 & (M+1/M)/2\end{bmatrix},$$

and for the A-polynomial,

$$\eta'(s)=\begin{bmatrix}M & 1\\ 0 & M^{-1}\end{bmatrix},\quad \eta'(t)=\begin{bmatrix}M & 0\\ t & M^{-1}\end{bmatrix}.$$

3.1. *The complex distance polynomial*

Since we are interested in the excellent component (the geometric component) of $R\left(\pi_1(X_{2n})\right)$, in this subsection we set $M=e^{\frac{i\alpha}{2}}$. Given the fundamental group of a twist knot

$$\pi_1(X_{2n})=\left\langle s,t \mid swt^{-1}w^{-1}=1\right\rangle,$$

where $w=(ts^{-1}t^{-1}s)^n$, let $S=\eta(s)$ and $T=\eta(t)$. Then the trace of S and the trace of T are both $2\cos\frac{\alpha}{2}$. Let ρ be the complex distance between the axes of S and T and set $W=\eta(w)$.

Lemma 3.1. *For $c\in \mathrm{SL}(2,\mathbb{C})$ which satisfies $cS=T^{-1}c$ and $c^2=-I$,*

$$SWT^{-1}W^{-1}=-(SWc)^2$$

Proof.

$$\begin{aligned}(SWc)^2&=SWcSWc=SWT^{-1}c(TS^{-1}T^{-1}S)^n c\\&=SWT^{-1}(S^{-1}TST^{-1})^n c^2=-SWT^{-1}W^{-1}.\end{aligned}$$ □

From the structure of the algebraic set of $R\left(\pi_1(X_{2n})\right)$ with coordinates $\eta(s)$ and $\eta(t)$ we have the defining equation of $R\left(\pi_1(X_{2n})\right)$. By plugging in $e^{\frac{i\alpha}{2}}$ into M of that equation and changing the variables to $B=\cos\frac{\alpha}{2}$ and $V=\cosh\rho$, we have the following theorem.

Theorem 3.2. *For $B=\cos\frac{\alpha}{2}$, $V=\cosh\rho$ is a root of the following complex distance polynomial $P_{2n}=P_{2n}(V,B)$ which is given recursively by*

$$P_{2n}=\begin{cases}\left(\left(4B^4-8B^2+4\right)V^2-4B^4+8B^2-2\right)P_{2(n-1)}-P_{2(n-2)} & \text{if } n>1\\ \left(\left(4B^4-8B^2+4\right)V^2-4B^4+8B^2-2\right)P_{2(n+1)}-P_{2(n+2)} & \text{if } n<-1\end{cases}$$

with initial conditions

$$P_{-2}(V,B) = \left(2B^2-2\right)V + 2B^2 - 1,$$
$$P_0(V,B) = 1,$$
$$P_2(V,B) = \left(4B^4 - 8B^2 + 4\right)V^2 + \left(2 - 2B^2\right)V - 4B^4 + 6B^2 - 1.$$

Proof. Note that $SWT^{-1}W^{-1} = I$, which gives the defining equations of $R(\pi_1(X_{2n}))$, is equivalent to $(SWc)^2 = -I$ in $\mathrm{SL}(2,\mathbb{C})$ by Lemma 3.1 and $(SWc)^2 = -I$ in $\mathrm{SL}(2,\mathbb{C})$ is equivalent to $\mathrm{tr}(SWc) = 0$.

We may assume

$$c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} \cos\frac{\alpha}{2} & ie^{\frac{\rho}{2}}\sin\frac{\alpha}{2} \\ ie^{-\frac{\rho}{2}}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}, \quad T = \begin{bmatrix} \cos\frac{\alpha}{2} & ie^{-\frac{\rho}{2}}\sin\frac{\alpha}{2} \\ ie^{\frac{\rho}{2}}\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix},$$

and let $U = TS^{-1}T^{-1}S$.

Then the equation,

$$\begin{aligned}\mathrm{tr}(SWc) &= \mathrm{tr}(SU^n c) = \mathrm{tr}(SU^{n-1}cU^{-1}) = \mathrm{tr}(SU^{n-1}c)\mathrm{tr}(U^{-1}) - \mathrm{tr}(SU^{n-1}cU) \\ &= \mathrm{tr}(SU^{n-1}c)\mathrm{tr}(U^{-1}) - \mathrm{tr}(SU^{n-2}c) = 0 \text{ if } n > 1\end{aligned}$$

or

$$\begin{aligned}\mathrm{tr}(SWc) &= \mathrm{tr}(SU^n c) = \mathrm{tr}(SU^{n+1}cU) = \mathrm{tr}(SU^{n+1}c)\mathrm{tr}(U) - \mathrm{tr}(SU^{n+1}cU^{-1}) \\ &= \mathrm{tr}(SU^{n+1}c)\mathrm{tr}(U) - \mathrm{tr}(SU^{n+2}c) = 0 \text{ if } n < -1,\end{aligned}$$

gives the complex distance polynomial, where the third equality comes from the Cayley-Hamilton theorem. By direct computations, $\mathrm{tr}(Sc)$, $\mathrm{tr}(SUc)$, and $\mathrm{tr}(SU^{-1}c)$ have $2i\sinh\frac{\rho}{2}\sin\frac{\alpha}{2}$ as a common factor. Hence, all of $\mathrm{tr}(SWc)$'s have $2i\sinh\frac{\rho}{2}\sin\frac{\alpha}{2}$ as a common factor. Actually, the common factor comes from the reducible representations. Just as the A-polynomials, we left the common factor out of our complex distance polynomials. We divide $\mathrm{tr}(SWc)$ by $2i\sinh\frac{\rho}{2}\sin\frac{\alpha}{2}$ and denote $\mathrm{tr}(SWc)/(2i\sinh\frac{\rho}{2}\sin\frac{\alpha}{2})$ by P_{2n}. We used *Mathematica* for the calculations. □

3.2. *A-polynomial*

Let $l = w^*w$ and $l_* = ww^*$, where w^* is the word obtained by reversing w. Then l and l_* are longitudes which are null-homologus in X_{2n}. We use l_* for this subsection and use both l and l_* in Section 4 to keep the original form in [13] and to keep the familar l_*. One can also deal with Section 4 with only l_* or l. Define R_U to be a subset of $R = \mathrm{Hom}(\pi_1(X_{2n}), \mathrm{SL}(2,\mathbb{C}))$ such that $\eta'(l_*)$ and $\eta'(s)$ are upper triangular. Since every representation can be conjugated into this form, any element of R is equivalent to an element of R_U. By adding the equation stating that the bottom-left entry of the matrix corresponding to $\eta'(l_*)$ is zero (the bottom-left entry of the matrix $\eta'(s)$ is already zero and the equation that the bottom-left entry of the matrix corresponding to $\eta'(l_*)$ is equal to zero is redundant in our setting), we have defining equations of R_U and hence R_U is an algebraic subset of R.

Define an *eigenvalue map*

$$\xi \equiv (\xi_{l_*} \times \xi_s) : R_U \longrightarrow \mathbb{C}^2$$

given by taking the top-left entries of $\eta'(l_*)$, L, and of $\eta'(s)$, M. The closure of the image $\xi(C)$ of an algebraic component C of R_U is an algebraic subset of $\mathbb{C}^2$. If the closure of the image $\xi(C)$ is a curve, there is a unique defining polynomial of this curve up to constant multiples. The *A-polynomial* of the knot T_{2n} is defined by the product of all defining polynomials of image curves of R_U. The A-polynomial of a knot can be defined up to sign [1].

Practically, if we let $r = r(M,t)$ be the upper right entry of $\eta'(sw) - \eta'(wt)$ and $q = q(M,t)$ be the upper left entry of $\eta'(l_*)$, then the A-polynomial of the knot T_{2n} can be obtained by taking the resultant of $M^{u_1} r$ and $M^{u_2}(q - L)$ over t, where the exponents u_1 and u_2 are chosen so that $M^{u_1} r$ and $M^{u_2}(q-L)$ become polynomials.

In [10, Theorem 1], Hoste and Shanahan presented the A-polynomial of the twist knots.

4. Pythagorean theorem

Let $L_\eta = \eta(l)$ and $L_{*\eta} = \eta(l_*)$. If we let $l_t = lt$ and $l_s = l_* s$, then $L_T = \eta(l_t) = L_\eta T$, $L_S = \eta(l_s) = L_{*\eta} S$ and we have the following lemma.

Lemma 4.1.

$$\begin{aligned} &\mathrm{tr}(S^{-1}L_T) = \mathrm{tr}(S^{-1}T) \textit{ if } n \geq 1 \textit{ and} \\ &\mathrm{tr}(T^{-1}L_S) = \mathrm{tr}(S^{-1}T) \textit{ if } n \leq -1. \end{aligned}$$

Proof. Since

$$\begin{aligned} S^{-1}L_T &= T^{-1}S^{-1}T(ST^{-1}S^{-1}T)^{n-1} \cdot (TS^{-1})(T^{-1}STS^{-1})^{n-1}T^{-1}ST \\ &= \left((T^{-1}STS^{-1})^{n-1}T^{-1}ST\right)^{-1}(TS^{-1})(T^{-1}STS^{-1})^{n-1}T^{-1}ST \text{ if } n \geq 1 \end{aligned}$$

and

$$\begin{aligned} T^{-1}L_S &= S^{-1}T^{-1}S(TS^{-1}T^{-1}S)^{n-1} \cdot (ST^{-1})(S^{-1}TST^{-1})^{n-1}S^{-1}TS \\ &= \left((S^{-1}TST^{-1})^{n-1}S^{-1}TS\right)^{-1}(ST^{-1})(S^{-1}TST^{-1})^{n-1}S^{-1}TS \text{ if } n \leq -1, \end{aligned}$$

we have

$$\begin{aligned} &\mathrm{tr}(S^{-1}L_T) = \mathrm{tr}(TS^{-1}) = \mathrm{tr}(S^{-1}T) \text{ if } n \geq 1 \text{ and} \\ &\mathrm{tr}(T^{-1}L_S) = \mathrm{tr}(ST^{-1}) = \mathrm{tr}(TS^{-1}) = \mathrm{tr}(S^{-1}T) \text{ if } n \leq -1. \end{aligned}$$

□

Definition 4.2. The *complex length* of the longitude l or l_* is the complex number γ_α modulo $2\pi\mathbb{Z}$ satisfying

$$\mathrm{tr}(\eta(l)) = \mathrm{tr}(\eta(l_*)) = 2\cosh\frac{\gamma_\alpha}{2}.$$

Note that $l_\alpha = |Re(\gamma_\alpha)|$ is the real length of the longitude of the cone-manifold $T_{2n}(\alpha)$.

We prepare and prove Theorem 4.3 for $T_{2n}(\alpha)$ with $n \geq 1$. For $n < -1$, the same Pythagrean theorem is obtained by replacing T and L_T with S and L_S. We will use the oriented line matrix corresponding to a given matrix. One can refer to [5, Section V] for oriented line matrices. Denote by $l(N)$ the line matrix corresponding to a matrix, N, in $\mathrm{SL}(2,\mathbb{C})$. Then $l(N) = (N - N^{-1})/\sqrt{det(N - N^{-1})}$.

By sending common fixed points of T and L_η to 0 and ∞, we have

$$T = \begin{bmatrix} e^{\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} \end{bmatrix}, \quad L_\eta = \begin{bmatrix} e^{\frac{\gamma_\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma_\alpha}{2}} \end{bmatrix},$$

$$L_T = L_\eta T = \begin{bmatrix} e^{\frac{\gamma_\alpha + i\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma_\alpha + i\alpha}{2}} \end{bmatrix},$$

and the following line matrices

$$\begin{aligned} l(T) &= \frac{T - T^{-1}}{2i\sinh\frac{i\alpha}{2}} \\ &= \frac{1}{i(e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}})} \begin{bmatrix} e^{\frac{i\alpha}{2}} - e^{-\frac{i\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\alpha}{2}} - e^{\frac{i\alpha}{2}} \end{bmatrix} \\ &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} l(L_T) &= \frac{L_T - L_T^{-1}}{2i\sinh\frac{\gamma_\alpha + i\alpha}{2}} \\ &= \frac{1}{i(e^{\frac{\gamma_\alpha + i\alpha}{2}} - e^{-\frac{\gamma_\alpha + i\alpha}{2}})} \begin{bmatrix} e^{\frac{\gamma_\alpha + i\alpha}{2}} - e^{-\frac{\gamma_\alpha + i\alpha}{2}} & 0 \\ 0 & e^{-\frac{\gamma_\alpha + i\alpha}{2}} - e^{\frac{\gamma_\alpha + i\alpha}{2}} \end{bmatrix} \\ &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \end{aligned}$$

which give the orientations of axes of T and L_T.

Figure 3 is the fundamental polyhedron for $T_2(\pi)$. The double branched covering space of the polyhedron along $\overline{P_3Q_0P_8}$ and $\overline{P_0Q_1P_5}$ is the lens space $L(5,3)$.

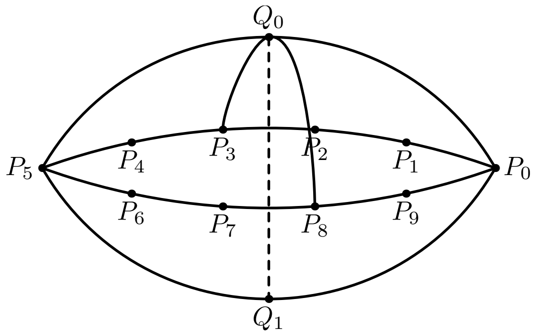

Fig. 3. Fundamental polyhedron for $4_1(\pi)$

The fundamental polyhedron for the hyperbolic cone-manifold of $T_2(\alpha)$ can be obtained from the fundamental polyhedron for $T_2(\pi)$ by deforming the cone-angle continuously. Recall that a Lambert quadrangle is a quadrangle with three right angles and one acute angle, not necessarily lying on a plane. You can consult [5, p. 83 of Section VI] or [13] for the trigonometry of a Lambert quadrangle or a right angled hexagon. Let m be the midpoint of $\overline{P_7P_8}$. Then the quadrangle $Q_0Q_1mP_8$ is a Lambert quadrangle with acute angle $\angle Q_0Q_1m = \alpha/2$, which can be considered as a right angled hexagon which is a generalized right angled triangle. The six sides are $\big(\rho,\ (\pi-\frac{\alpha}{2})i,\ *_1,\ \frac{\pi}{2}i,\ *_2,\ \frac{\gamma_\alpha}{4}+\frac{i\alpha}{2}\big)$. By applying the Law of Cosines to the hexagon, we get the formula in the following theorem geometrically and the same argument works for all twist knots. Hence, we call the following theorem Pythagorean Theorem.

Now, we are ready to prove the following theorem which gives Theorem 4.4. Recall that γ_α modulo $2\pi\mathbb{Z}$ is the *complex length* of the longitude l or l_* of $T_{2n}(\alpha)$.

Theorem 4.3. *(Pythagorean Theorem) Let $T_{2n}(\alpha)$ be a hyperbolic cone-manifold and let ρ be the complex distance between the oriented axes S and T. Then we have*

$$i\cosh\rho = \cot\frac{\alpha}{2}\tanh(\frac{\gamma_\alpha}{4}+\frac{i\alpha}{2}).$$

Proof. Suppose $n \geq 1$.

$$\begin{aligned}
\cosh\rho &= -\frac{\mathrm{tr}(l(S)l(T))}{2} \\
&= -\frac{\mathrm{tr}(l(S)l(L_T))}{2} \\
&= \frac{\mathrm{tr}((S-S^{-1})(L_T-L_T^{-1}))}{8\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha+i\alpha}{2}} \\
&= \frac{\mathrm{tr}(SL_T - S^{-1}L_T - SL_T^{-1} + (L_TS)^{-1}))}{8\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha+i\alpha}{2}} \\
&= \frac{2(\mathrm{tr}(SL_T) - \mathrm{tr}(S^{-1}L_T))}{8\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha+i\alpha}{2}} \\
&= \frac{\mathrm{tr}(S)\mathrm{tr}(L_T) - 2\mathrm{tr}(S^{-1}L_T)}{4\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha+i\alpha}{2}} \\
&= \frac{\mathrm{tr}(S)\mathrm{tr}(L_T) - 2\mathrm{tr}(S^{-1}T)}{4\sinh\frac{i\alpha}{2}\sinh\frac{\gamma_\alpha+i\alpha}{2}}
\end{aligned}$$

where the first equality comes from [5, p. 68], the sixth equality comes from the Cayley-Hamilton theorem, and the seventh equality comes from Lemma 4.1.

Let $\nu = \frac{\alpha}{2}$, $\Lambda = \frac{\gamma_\alpha+i\alpha}{2}$, and $V = \cosh\rho$. Then $\mathrm{tr}(S) = 2\cos\nu$, $\mathrm{tr}(L_T) = 2\cosh\Lambda$, and

$$\begin{aligned}
\mathrm{tr}(S^{-1}T) &= tr\left(\begin{pmatrix} \cos\nu & -ie^{\frac{\rho}{2}}\sin\nu \\ -ie^{-\frac{\rho}{2}}\sin\nu & \cos\nu \end{pmatrix}\begin{pmatrix} \cos\nu & ie^{-\frac{\rho}{2}}\sin\nu \\ ie^{\frac{\rho}{2}}\sin\nu & \cos\nu \end{pmatrix}\right) \\
&= \cos^2\nu + e^{\rho}\sin^2\nu + e^{-\rho}\sin^2\nu + \cos^2\nu \\
&= 2(\cos^2\nu + V\sin^2\nu).
\end{aligned}$$

Hence,

$$V = \frac{4\cos\nu\cosh\Lambda - 4(\cos^2\nu + V\sin^2\nu)}{4i\sin\nu\sinh\Lambda}$$

which is equivalent to

$$V\sin\nu(\sin\nu + i\sinh\Lambda) = \cos\nu(\cosh\Lambda - \cos\nu).$$

By solving for V, we have

$$\begin{aligned}
V &= \cot\nu \frac{\cosh\Lambda - \cos\nu}{\sin\nu + i\sinh\Lambda} \\
&= \cot\nu \frac{\cosh\Lambda - \cosh i\nu}{i\sinh\Lambda - i\sinh i\nu} \\
&= -i\cot\nu \frac{\cosh\Lambda - \cosh i\nu}{\sinh\Lambda - \sinh i\nu} \\
&= -i\cot\nu \frac{2\sinh(\frac{\Lambda+i\nu}{2})\sinh(\frac{\Lambda-i\nu}{2})}{2\cosh(\frac{\Lambda+i\nu}{2})\sinh(\frac{\Lambda-i\nu}{2})} \\
&= -i\cot\nu\tanh\frac{\Lambda+i\nu}{2}.
\end{aligned}$$

By putting back $\Lambda = \frac{\gamma_\alpha + i\alpha}{2} = \frac{\gamma_\alpha}{2} + i\nu$, we have

$$V = -i\cot\nu\tanh(\frac{\gamma_\alpha}{4} + i\nu)$$

which is equivalent to

$$i\cosh\rho = \cot\frac{\alpha}{2}\tanh(\frac{\gamma_\alpha}{4} + \frac{i\alpha}{2}). \qquad \square$$

Pythagorean theorem 4.3 gives the following theorem which relates the zeros of $A_{2n}(L,M)$ and the zeros of $P_{2n}(V,B)$ for $M = e^{\frac{i\alpha}{2}}$, $B = \cos\frac{\alpha}{2}$ and $A = \cot\frac{\alpha}{2}$.

Theorem 4.4. *Let $A = \cot\frac{\alpha}{2}$ and $M = e^{\frac{i\alpha}{2}}$. Then the following formulae show that there is a one to one correspondence between the zeros of $A_{2n}(L,M)$ and the zeros of $P_{2n}(V,B)$:*

$$iV = A\frac{LM^2-1}{LM^2+1} \text{ and } L = M^{-2}\frac{A+iV}{A-iV}.$$

Proof. With the same notation as in the proof of Theorem 4.3,

$$\begin{aligned}
iV &= i\cosh\rho \\
&= \cot\nu\tanh(\frac{\gamma_\alpha}{4} + i\nu) \\
&= \cot\nu\frac{\sinh(\frac{\gamma_\alpha}{4} + i\nu)}{\cosh(\frac{\gamma_\alpha}{4} + i\nu)} \\
&= \cot\nu\frac{e^{\frac{\gamma_\alpha}{4}+i\nu} - e^{-(\frac{\gamma_\alpha}{4}+i\nu)}}{e^{\frac{\gamma_\alpha}{4}+i\nu} + e^{-(\frac{\gamma_\alpha}{4}+i\nu)}} \\
&= \cot\frac{\alpha}{2}\frac{e^{\frac{\gamma_\alpha}{2}+i\alpha} - 1}{e^{\frac{\gamma_\alpha}{2}+i\alpha} + 1} \\
&= A\frac{LM^2-1}{LM^2+1}.
\end{aligned}$$

If we solve the above equation,

$$iV = A\frac{LM^2-1}{LM^2+1},$$

for L, we have

$$L = M^{-2}\frac{A+iV}{A-iV}. \qquad \square$$

5. Proof of Theorem 1.1

We mention here that the proof can be done without referring to A-polynomial. We identified L with a root of $A_{2n}(M,L)$ because A-polynomial is rather well-known.

For $n \geq 1$ and $M = e^{i\frac{\alpha}{2}}$ ($B = \cos\frac{\alpha}{2}$), $A_{2n}(M,L)$ and $P_{2n}(V,A)$ have $2n$ component zeros, and for $n < -1$, $-(2n+1)$ component zeros. For each n, there exists an angle $\alpha_0 \in [\frac{2\pi}{3},\pi)$ such that $T_{2n}(\alpha)$ is hyperbolic for $\alpha \in (0,\alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0,\pi]$ [18,8,11,19]. From the following Equality 5.1, when $|L| = 1$, which is equivalent to $\alpha = \alpha_0$, $\mathrm{Im}(V) = 0$. Hence, when α increases from 0 to α_0, two complex numbers V and $\overline{V}$ approach to a same real number. In other words, $P_{2n}(V,\cos\frac{\alpha_0}{2})$ has a muliple root and hence $A_{2n}(L,e^{\frac{i\alpha_0}{2}})$ has a multiple root by Theorem 4.4. Denote by $D(T_{2n}(\alpha))$ be the greatest common factor of the discriminant of $A_{2n}(L,e^{\frac{i\alpha}{2}})$ over L and the discriminant of $P_{2n}(V,\cos\frac{\alpha}{2})$ over V. Then α_0 will be one of the zeros of $D(T_{2n}(\alpha))$.

From Theorem 4.4, we have following equality,

$$|L|^2 = \left|\frac{A+iV}{A-iV}\right|^2 = \frac{|A|^2+|V|^2-2AIm(V)}{|A|^2+|V|^2+2AIm(V)}. \tag{5.1}$$

For the volume, we can either choose $|L| \geq 1$ or $|L| \leq 1$. We choose L with $|L| \geq 1$ and hence we have $\mathrm{Im}(V) \leq 0$ by Equality 5.1. Using the Schläfli formula, we calculate the volume of $\eta(T_{2n}) = \eta(T_{2n}(0))$ for each component with $|L| \geq 1$ and having one of the zeros α_0 of $D(T_{2n}(\alpha))$ with $\alpha_0 \in [\frac{2\pi}{3},\pi)$ on it. The component which gives the maximal volume is the excellent component [4,6]. On the geometric component we have the volume of a hyperbolic cone-manifold $T_{2n}(\alpha)$ for $0 \leq \alpha < \alpha_0$:

$$\begin{aligned}
\mathrm{Vol}(T_{2n}(\alpha)) &= -\int_{\alpha_0}^{\alpha} \frac{l_\alpha}{2}\, d\alpha \\
&= -\int_{\alpha_0}^{\alpha} \log|L|\, d\alpha \\
&= -\int_{\pi}^{\alpha} \log|L|\, d\alpha \\
&= \int_{\alpha}^{\pi} \log|L|\, d\alpha \\
&= \int_{\alpha}^{\pi} \log\left|\frac{A+iV}{A-iV}\right| d\alpha,
\end{aligned}$$

where the first equality comes from the Schläfli formula for cone-manifolds (Theorem 3.20 of [2]), the second equality comes from the fact that $l_\alpha = |Re(\gamma_\alpha)|$ is the real length of the longitude of $T_{2n}(\alpha)$, the third equality comes from the fact that $\log |L| = 0$ for $\alpha_0 < \alpha \leq \pi$ by Equality 5.1 since all the characters are real (the proof of Proposition 6.4 of [19]) for $\alpha_0 < \alpha \leq \pi$, and $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ is a zero of the discriminant $D(T_{2n}(\alpha))$.

We note that the fundamental set of the two-bridge link orbifolds are constructed in [14]. We also note that the explicit formulae for the Chern-Simons invariant of the twist knot orbifolds are presented in [7] and the A-polynomials of twist knots are obtained from the complex distance polynomials in [7].

6. Volumes of the hyperbolic twist knot cone-manifolds and of its cyclic coverings

Table 1 gives the approximate volume of $\eta(T_{2n})$ for each n between -9 and 9 except the unknot and the torus knot and for each component with $\mathrm{Im}(V) \leq 0$ and having one of the zeros of $D(T_{2n}(\alpha))$ with $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ on it. We used Simpson's rule for the approximation with 10^4 intervals from 0 to α_0. In that way our approximate volume on the geometric component is the same as that of SnapPea up to four decimal points. The geometric volume is written one more time on the rightmost column.

Table 2 (resp. Table 3) gives the approximate volume of the hyperbolic twist knot cone-manifold, $V\left(T_{2n}(\frac{2\pi}{k})\right)$ for n between 1 and 9 (resp. for n between -9 and -2) and for k between 3 and 10, and of its cyclic covering, $V\left(M_k(T_{2n})\right)$. We again used Simpson's rule for the approximation with 10^4 intervals from $\frac{2\pi}{k}$ to α_0. We used *Mathematica* for the calculations.

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Table 1: The volume of $\eta(T_{2n})$ for each component with $\mathrm{Im}(V) \leq 0$ and having one of the zeros of $D(T_{2n}(\alpha))$ with $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ on it. N refers to the number of zeros of $D(T_{2n}(\alpha))$ in $[\frac{2\pi}{3}, \pi)$ and Z refers to the zeros of $D(T_{2n}(\alpha))$ in $[\frac{2\pi}{3}, \pi)$. The volume of T_{2n}, $V(T_{2n})$, is written one more time on the rigthtmost column.

2n	N	Z	$V(\eta(T_{2n}))$	$V(T_{2n})$
2	1	2.0944	2.02988	2.02988
4	1	2.57414	3.16396	3.16396
6	1	2.75069	3.4272	3.4272
8	2	2.84321	3.52619	3.52619
		2.3287	3.09308	
10	2	2.90026	3.57388	3.57388
		2.48721	3.29551	
12	3	2.93897	3.60046	3.60046
		2.59356	3.40614	
		2.22905	3.06705	
14	3	2.96697	3.61679	3.61679
		2.67	3.47332	
		2.35895	3.22608	
16	4	2.98817	3.62753	3.62753
		2.72765	3.5172	
		2.45606	3.3286	
		2.1754	3.05359	
18	4	3.00477	3.63497	3.63497
		2.7727	3.54747	
		2.53152	3.39869	
		2.28381	3.18371	

2n	N	Z	$V(\eta(T_{2n}))$	$V(T_{2n})$
-2	1	a torus knot		
-4	1	2.40717	2.82812	2.82812
-6	1	2.67879	3.33174	3.33174
-8	2	2.80318	3.48666	3.48666
		2.21583	2.92126	
-10	2	2.87475	3.55382	3.55382
		2.41665	3.21098	
-12	3	2.9213	3.58891	3.58891
		2.54513	3.35826	
		2.14593	2.95204	
-14	3	2.95401	3.60954	3.60954
		2.63466	3.44354	
		2.29908	3.15591	
-16	4	2.97825	3.62268	3.62268
		2.70071	3.49742	
		2.41076	3.28251	
		2.10986	2.96721	
-18	4	2.99694	3.63157	3.63157
		2.75147	3.53365	
		2.49601	3.36675	
		2.2329	3.12459	

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Table 2: Volume of the hyperbolic twist knot cone-manifold, $V\left(T_{2n}(\frac{2\pi}{k})\right)$ for n between 1 and 9 and for k between 1 and 10, and of its cyclic covering, $V\left(M_k(T_{2n})\right)$.

k	$V\left(T_2(\frac{2\pi}{k})\right)$	$V\left(M_k(T_2)\right)$
3	Euclidean	
4	0.507471	2.02988
5	0.937207	4.68603
6	1.22129	7.32772
7	1.41175	9.88228
8	1.54386	12.3509
9	1.6386	14.7474
10	1.70857	17.0857

k	$V\left(T_4(\frac{2\pi}{k})\right)$	$V\left(M_k(T_4)\right)$
3	0.654246	1.96274
4	1.64974	6.59895
5	2.1789	10.8945
6	2.47479	14.8488
7	2.65528	18.587
8	2.77325	22.186
9	2.85453	25.6908
10	2.91289	29.1289

k	$V\left(T_6(\frac{2\pi}{k})\right)$	$V\left(M_k(T_6)\right)$
3	1.13433	3.40299
4	2.1114	8.4456
5	2.5728	12.864
6	2.82779	16.9667
7	2.98374	20.8862
8	3.08602	24.6882
9	3.15669	28.4102
10	3.20752	32.0752

k	$V\left(T_8(\frac{2\pi}{k})\right)$	$V\left(M_k(T_8)\right)$
3	1.39476	4.18428
4	2.29275	9.17098
5	2.72019	13.6009
6	2.95903	17.7542
7	3.10589	21.7413
8	3.20251	25.6201
9	3.26938	29.4244
10	3.31754	33.1754

k	$V\left(T_{10}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{10})\right)$
3	1.52984	4.58951
4	2.37879	9.51514
5	2.79028	13.9514
6	3.02165	18.1299
7	3.16431	22.1502
8	3.25831	26.0664
9	3.32342	29.9108
10	3.37035	33.7035

k	$V\left(T_{12}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{12})\right)$
3	1.60485	4.81454
4	2.42618	9.70474
5	2.82906	14.1453
6	3.05637	18.3382
7	3.19675	22.3773
8	3.28932	26.3145
9	3.35347	30.1812
10	3.39973	33.9973

k	$V\left(T_{14}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{14})\right)$
3	1.65032	4.95096
4	2.45507	9.82028
5	2.85276	14.2638
6	3.07763	18.4658
7	3.21663	22.5164
8	3.30832	26.4666
9	3.37191	30.3472
10	3.41775	34.1775

k	$V\left(T_{16}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{16})\right)$
3	1.67992	5.03976
4	2.47398	9.89592
5	2.86831	14.3415
6	3.09158	18.5495
7	3.22967	22.6077
8	3.3208	26.5664
9	3.38401	30.4561
10	3.42959	34.2959

k	$V\left(T_{18}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{18})\right)$
3	1.70026	5.10079
4	2.48704	9.94814
5	2.87906	14.3953
6	3.10124	18.6074
7	3.23869	22.6708
8	3.32948	26.6358
9	3.39265	30.5339
10	3.43779	34.3779

Table 3: Volume of the hyperbolic twist knot cone-manifold, $V\left(T_{2n}(\frac{2\pi}{k})\right)$ for n between -9 and -2 and for k between 1 and 10, and of its cyclic covering, $V\left(M_k(T_{2n})\right)$.

k	$V\left(T_{-4}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{-4})\right)$
3	0.314236	0.942707
4	1.18737	4.7495
5	1.72248	8.61241
6	2.04253	12.2552
7	2.24401	15.7081
8	2.37774	19.022
9	2.47065	22.2358
10	2.53766	25.3766

k	$V\left(T_{-6}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{-6})\right)$
3	0.927278	2.78183
4	1.93717	7.74869
5	2.42921	12.1461
6	2.70001	16.2001
7	2.865	20.055
8	2.97296	23.7837
9	3.04743	27.4269
10	3.10095	31.0095

k	$V\left(T_{-8}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{-8})\right)$
3	1.28595	3.85786
4	2.22061	8.88245
5	2.6616	13.308
6	2.9068	17.4408
7	3.05724	21.4007
8	3.15609	25.2487
9	3.22445	29.0201
10	3.27366	32.7366

k	$V\left(T_{-10}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{-10})\right)$
3	1.47286	4.41857
4	2.34274	9.37094
5	2.76087	13.8044
6	2.99535	17.9721
7	3.13977	21.9784
8	3.23486	25.8789
9	3.3007	29.7063
10	3.34815	33.4815

k	$V\left(T_{-12}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{-12})\right)$
3	1.57236	4.71709
4	2.40564	9.62256
5	2.81223	14.0612
6	3.0413	18.2478
7	3.18267	22.2787
8	3.27585	26.2068
9	3.34042	30.0638
10	3.38696	33.8696

k	$V\left(T_{-14}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{-14})\right)$
3	1.63018	4.89055
4	2.44226	9.76903
5	2.84224	14.2112
6	3.06819	18.4091
7	3.2078	22.4546
8	3.29988	26.3991
9	3.36371	30.2734
10	3.40974	34.0974

k	$V\left(T_{-16}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{-16})\right)$
3	1.66659	4.99977
4	2.46545	9.8618
5	2.86129	14.3065
6	3.08528	18.5117
7	3.22379	22.5665
8	3.31517	26.5213
9	3.37856	30.407
10	3.42425	34.2425

k	$V\left(T_{-18}(\frac{2\pi}{k})\right)$	$V\left(M_k(T_{-18})\right)$
3	1.69098	5.07294
4	2.48107	9.92429
5	2.87415	14.3707
6	3.09683	18.581
7	3.23457	22.642
8	3.32549	26.6039
9	3.38869	30.4982
10	3.43405	34.3405

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